MA20222 Problem Sheet 9

Do all questions and hand in your answers to the ⋆**starred**⋆ questions as instructed by your tutor.

⋆E9.1. For the matrices T corresponding to the Jacobi and Gauss–Seidel iteration in E8.2, determine the spectral radius $\rho(T)$. Which of the iterations converge?

⋆E9.2. Suppose that T is a $d \times d$ matrix with $\|T\|_{\text{op}} < 1$, for some operator norm $\|T\|_{\text{op}}$ with respect to a vector norm $\|\boldsymbol{x}\|$.

(a) Let I denote the $d \times d$ identity matrix. Given that the expansion of the inverse matrix $(I-T)^{-1} = I + T + T^2 + \cdots +$ converges, prove that

$$\|(I-T)^{-1}\|_{\text{op}} \leq \frac{1}{1-\|T\|_{\text{op}}}.$$

(b) For a vector $\boldsymbol{c} \in \mathbb{R}^n$, let $\boldsymbol{x} \in \mathbb{R}^n$ satisfy $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$ and consider the iteration $\boldsymbol{x}_{n+1} = T\boldsymbol{x}_n + \boldsymbol{c}$. Show that

$$\|\boldsymbol{x}_n - \boldsymbol{x}\| \leq \frac{\|T\|_{\text{op}}^n}{1-\|T\|_{\text{op}}} \|\boldsymbol{x}_1 - \boldsymbol{x}_0\|. \qquad (\star)$$

Hint: use the following estimate from class:

$$\|\boldsymbol{x}_n - \boldsymbol{x}\| \leq \|T\|_{\text{op}}^n \|\boldsymbol{x}_0 - \boldsymbol{x}\|. \qquad (\dagger)$$

(c) Why is $(\star)$ more useful in applications that $(\dagger)$?

E9.3. Let $A = D + L + U$ be split into diagonal and lower- and upper-triangular parts. Let T_{J} denote the Jacobi iteration matrix and T_{GS} the Gauss–Seidel iteration matrix.

(a) Show that λ is an eigenvalue of T_{J} if and only if $\det|-\lambda D - L - U| = 0$.

(b) Show that λ is an eigenvalue of T_{GS} if and only if $\det|-\lambda(D+L) - U| = 0$.

(c) Assume that, for $a, b \neq 0$,

$$\det|aD - L - U| = \det|aD - b^{-1}L - bU|. \qquad (*)$$

Show that λ is an eigenvalue of T_{GS} if and only $\lambda^{1/2}$ is an eigenvalue of T_{J}.

E9.4. Formulate the Jacobi iteration as $\boldsymbol{x}_{k+1} = T\boldsymbol{x}_k + \boldsymbol{c}$ for the $d \times d$ finite-difference matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

(a) Show that $\boldsymbol{u}_k = (\sin k\pi h, \sin 2k\pi h, \ldots, \sin dk\pi h)^{\mathsf{T}}$, for $h = 1/(d+1)$ and $k = 1, \ldots, d$, is an eigenvector of T. Use this to prove the Jacobi iteration converges for this class of matrices.

(b) Show that $(*)$ holds for the finite-difference matrix A.

(c) Which of the iterations (Jacobi and Gauss-Seidel) would converge faster for this matrix?