MA20222 Problem Sheet 8

Do all questions and hand in your answers to the ⋆**starred**⋆ questions as instructed by your tutor.

E8.1. Let U be a non-singular upper triangular $d \times d$ matrix such that $U_{ij} = 0$ for $i > j$ and $U_{ii} \neq 0$. Consider the work involved in solving the system of d linear equations $U\boldsymbol{x} = \boldsymbol{b}$ by backward substitution, where $\boldsymbol{b} \in \mathbb{R}^d$ is given and $\boldsymbol{x} \in \mathbb{R}^d$ is to be found. That is, we evaluate

$$x_d = \frac{b_d}{U_{dd}} \quad \text{and} \quad x_i = \left(b_i - \sum_{j=i+1}^{d} U_{ij} x_j \right) \frac{1}{U_{ii}}, \quad i = d-1, d-2, \dots, 1.$$

(a) Assuming that each addition, subtraction, multiplication or division counts as one (arithmetic) operation, count how many operations are required to compute x_i, for each i.

(b) Hence establish how many operations are required in total to find $\boldsymbol{x}$ and thus establish the *complexity* of the backward-substitution algorithm.

⋆E8.2. Consider the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5. \end{aligned}$$

Formulate the Jacobi and Gauss–Seidel iterations for this system. Perform one step of each iteration with the initial guess $\boldsymbol{x}_0 = [0, 1, 0]^\mathsf{T}$.

E8.3. Write down the definitions of the norms $\|\cdot\|_\infty$, $\|\cdot\|_1$, and $\|\cdot\|_2$ for a vector $\boldsymbol{x} \in \mathbb{R}^n$.

Compute $\|\boldsymbol{x}\|_\infty$, $\|\boldsymbol{x}\|_1$, and $\|\boldsymbol{x}\|_2$ for $\boldsymbol{x} = [1, -2]^\mathsf{T}$ and $\boldsymbol{x} = [-1, 2, -3]^\mathsf{T}$.

⋆E8.4. a) Show that $\|\boldsymbol{x}\| := \max_{i=1,\dots,d} x_i$ is not a vector norm on $\mathbb{R}^d$.

b) Show that $\|\boldsymbol{x}\| = \left(\sum_{i=1}^d \sqrt{|x_i|}\right)^2$ is not a vector norm on $\mathbb{R}^d$.

⋆E8.5. For $d \times d$ matrices A and B, show conditions (a–d) for operator matrix norms by using properties of vector norms. Thus completing the proof of Theorem 5.1.

E8.6. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad B = A^\mathsf{T}.$$

Compute $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for A and B.

E8.7. From the definition

$$\|A\|_1 = \max_{\|\boldsymbol{x}\|_1 = 1} \|A\boldsymbol{x}\|_1,$$

show that $\|A\|_1 \leq \max_{1 \leq j \leq d} \sum_{i=1}^d |a_{ij}|$.

Now choose the vector $\boldsymbol{x} = (0, \dots, 1, 0, \dots, 0)^\mathsf{T}$, with a 1 in the kth position where

$$\sum_{i=1}^{d} |a_{ik}| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|,$$

to deduce that $\|A\|_1 = \max_{1 \leq j \leq d} \sum_{i=1}^d |a_{ij}|$.