MA20222 Problem Sheet 10

Do all questions and hand in your answers to the ⋆**starred**⋆ questions as instructed by your tutor.

E10.1. (a) Compute $\mathrm{Cond}_1(A)$ and $\mathrm{Cond}_\infty(A)$, the 1- and ∞-norm condition numbers of

$$A = \begin{bmatrix} 0.01 & 2 \\ 0 & 1 \end{bmatrix}.$$

(b) Compute $\|A\|_2$, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Let A be a diagonal matrix with entries $d_1, d_2, \dots, d_n$. Give an expression for $\mathrm{Cond}_2(A)$.

E10.2. (a) A scalar λ is an eigenvalue of a $d \times d$ matrix A if there exists a vector $\boldsymbol{u} \neq \boldsymbol{0}$ such that $A\boldsymbol{u} = \lambda\boldsymbol{u}$. Show that $|\lambda| \leq \|A\|_{\mathrm{op}}$ for any operator norm and any eigenvalue λ of A.

(b) Show that $\mathrm{Cond}(A) \geq 1$, where $\mathrm{Cond}(\cdot)$ is the condition number relative to an operator norm $\|\cdot\|_{\mathrm{op}}$.

(c) An orthogonal matrix Q is a transformation that leaves the Euclidean distance constant, in other words $\|Q\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ for each vector $\boldsymbol{x}$. Show that $\|Q\|_2 = 1$ for orthogonal Q and that $\mathrm{Cond}_2(Q) = 1$. (Orthogonal matrices are perfectly conditioned.)

⋆E10.3. Consider vectors $\boldsymbol{x}$, $\boldsymbol{\Delta x}$, $\boldsymbol{b}$, and $\boldsymbol{\Delta b}$ such that

$$A\boldsymbol{x} = \boldsymbol{b}, \qquad A\Delta\boldsymbol{x} = \Delta\boldsymbol{b}.$$

(a) By using $\|A\boldsymbol{x}\| \leq \|A\|_{\mathrm{op}} \|\boldsymbol{x}\|$, show that

$$\frac{\|\boldsymbol{\Delta x}\|}{\|\boldsymbol{x}\|} \leq \mathrm{Cond}(A) \frac{\|\boldsymbol{\Delta b}\|}{\|\boldsymbol{b}\|}.$$

(b) By using the above, estimate $\mathrm{Cond}(A)$ with respect to $\|\cdot\|_\infty$ in the case

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}, \qquad \boldsymbol{\Delta x} = \begin{bmatrix} 0.4 \\ -0.9 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} -0.9 \\ -5 \end{bmatrix}.$$