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WAVENUMBER-EXPLICIT BOUNDS IN TIME-HARMONIC ACOUSTIC SCATTERING*

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Abstract. We prove wavenumber-explicit bounds on the Dirichlet-to-Neumann map for the Helmholtz equation in the exterior of a bounded obstacle when one of the following three conditions holds: (i) the exterior of the obstacle is smooth and nontrapping, (ii) the obstacle is a nontrapping polygon, or (iii) the obstacle is star-shaped and Lipschitz. We prove bounds on the Neumann-to-Dirichlet map when condition (i) and (ii) hold. We also prove bounds on the solutions of the interior and exterior impedance problems when the obstacle is a general Lipschitz domain. These bounds are the sharpest yet obtained (for their respective problems) in terms of their dependence on the wavenumber. One motivation for proving these collection of bounds is that they can then be used to prove wavenumber-explicit bounds on the inverses of the standard second-kind integral operators used to solve the exterior Dirichlet, Neumann, and impedance problems for the Helmholtz equation.

Key words. Helmholtz equation, Dirichlet-to-Neumann, Neumann-to-Dirichlet, impedance boundary condition, wavenumber-explicit, semiclassical, boundary integral operator

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1. Introduction.

Proving bounds on solutions of the Helmholtz equation

$$\Delta u + k^2 u = -f \tag{1.1}$$

(where f is a given function and $k > 0$ is the wavenumber) is a classic problem. When a Helmholtz boundary value problem (BVP) has a unique solution, the solution can be bounded in terms of the data using Fredholm theory, since the variational, or weak, formulations of Helmholtz BVPs satisfy Gårding inequalities. The resulting bounds, however, are *not* explicit in the wavenumber k.

Obtaining k-explicit bounds on the Helmholtz equation has a long history, and we discuss some of this previous work in detail below. We mention at this stage the fundamental k-explicit bounds of Morawetz [44] and Vainberg [58] on the inverse of the Helmholtz operator in exterior domains that are nontrapping. The former bounds rely on certain identities for solutions of the Helmholtz equation, and the latter bounds are proved using much more general arguments that exploit the fact that the Helmholtz equation arises by taking the Fourier transform in time of the wave equation and then use the propagation of singularities results of Melrose and Sjöstrand [36], [37]. Since the inverse of the Helmholtz operator is the resolvent of the Laplacian, these bounds are often called resolvent estimates.

Given this area's long history, one might think that there are no more outstanding problems to solve. However, there has been a revival of interest in k-explicit bounds on solutions of the Helmholtz equation, largely motivated by the current interest in the k-explicit numerical analysis of wave propagation problems (see, e.g., the recent review articles [11], [16], [17], [18]), and this renewed interest has highlighted that several fundamental problems remain open.

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In particular,

(a) although the classic resolvent estimates of Morawetz and Vainberg in exterior nontrapping domains are sharp in their k-dependence, there do not yet exist sharp bounds on the Dirichlet-to-Neumann (DtN) and Neumann-to-Dirichlet (NtD) maps in these domains;

(b) there are relatively few bounds available for exterior problems in nonsmooth domains (mainly because the propagation of singularities on these domains is highly nontrivial);

(c) there do not yet exist sharp bounds on the solution of the interior impedance problem posed in a general Lipschitz domain.

Regarding (a). Although the classic resolvent estimates can be converted into bounds on the DtN and NtD maps (and this was done recently by Lakshtanov and Vainberg in [28]), the bounds obtained so far via this method appear not to be sharp in their k-dependence (and we prove this in this paper). Although these DtN and NtD bounds are of interest in their own right, they play an essential role in bounding the inverses of the integral operators used to solve the exterior Dirichlet and Neumann problems (see section 1.3).

Regarding (b). The resolvent estimates obtained by Morawetz in smooth domains can be extended to hold in nonsmooth star-shaped domains, since these estimates rely on identities that hold in Lipschitz domains. (See section 3.1 and [12, Lemma 3.8] for more details.) The more general arguments of Vainberg rely on results about propagation of singularities, and the relevant results for nonsmooth domains have only recently been obtained (see [40], [38], [59], [39], [8], and section 3.1).

Regarding (c). Many investigations of numerical methods for solving the Helmholtz equation begin by considering the Helmholtz equation in a bounded domain (to avoid the complications associated with imposing the radiation condition numerically). To obtain a BVP that is well-posed for every $k > 0$, an impedance boundary condition

$$\frac{\partial u}{\partial n} - \mathrm{i}\eta u = g \tag{1.2}$$

is applied, where g is a given function and η is a real constant. Because this interior impedance problem is used as a model problem for numerical analysis of the Helmholtz equation, several authors over the years have obtained bounds on the solution in terms of the data that are explicit in k and η [20], [33], [15], [18] (with [24], [7], and [30] considering closely related Helmholtz BVPs and [25], [41] considering the analogous BVP for the time-harmonic Maxwell equations). However, there do not yet exist sharp bounds (in terms of k- and η-dependence) on the solution of this BVP posed in a general Lipschitz domain.

Aside from its use as a model problem for numerical analysis, the interior impedance problem plays a fundamental role in the conditioning of the integral operators that are used to solve *exterior* problems. Indeed, to bound the inverses of the integral operators used to solve the exterior Dirichlet, Neumann, and impedance problems, one needs not only bounds on the exterior DtN, NtD, and impedance-to-Dirichlet maps but also a bound on the interior impedance problem. (If the reader is not familiar with boundary integral equations, then this may appear strange; however, each of the integral operators for the three exterior problems can also be used to solve the interior impedance problem. Therefore, it is natural that the norms of the inverses of the integral operators should depend on both the exterior and the interior problems.)

In this paper we do the following:

1. We prove bounds on the exterior DtN map, which are sharper in their k-dependence than any previously obtained bounds, when one of the following three conditions holds:
   (i) the exterior of the obstacle is a C^∞ nontrapping domain in two dimensions (2-d) or three dimensions (3-d),
   (ii) the obstacle is a nontrapping polygon (in 2-d),
   (iii) the obstacle is a star-shaped, Lipschitz domain in 2- or 3-d.

   We also prove bounds on the exterior NtD map in cases (i) and (ii), with the bounds for case (ii) being the first bounds on the NtD map for nonsmooth domains. (These DtN and NtD bounds therefore partially address the open problems (a) and (b) above.)
2. We prove bounds on the interior impedance problem in a general Lipschitz domain that are sharper in their k- and η-dependence than any previously obtained bounds (thus partially addressing the open problem (c) above). This method of proof also yields bounds on the exterior impedance problem.

Regarding 1. For the class of domains in (i), Lakshtanov and Vainberg [28] recently obtained DtN and NtD bounds in the trace spaces using the classic resolvent estimates. We use the same idea, but we sharpen the DtN bound in the trace spaces by a factor of $k^{3/2}$ and also prove DtN and NtD bounds when $\partial u/\partial n \in L^2(\Gamma)$ and $u \in H^1(\Gamma)$, where Γ denotes the boundary of the obstacle. (This case is particularly important for the applications of these bounds to integral equations; see section 1.3.) For the class of domains in (ii), we obtain the DtN and NtD bounds from the resolvent estimates in these domains recently obtained by Baskin and Wunsch [8] using results about the propagation of singularities in this type of domain. For the class of domains in (iii), a resolvent estimate for the Dirichlet problem was obtained by Chandler-Wilde and Monk in [12], essentially using the identities of Morawetz (see the discussion in section 3.1). The same argument used to prove bounds on the DtN map for the class of domains (ii) can then be used to prove bounds on the DtN map for the class (iii). By considering the specific cases of the circle and sphere and using results about the asymptotics of Bessel and Hankel functions, we are able to determine exactly how far from being sharp (in terms of k-dependence) the bounds for the classes of domains (i) and (iii) are.

Regarding 2. The impedance boundary condition is somewhat different from the Dirichlet and Neumann boundary conditions in that, for the time-dependent problem, it means that energy is either emitted or absorbed by the boundary (depending on the sign of η) and thus is not conserved as in the Dirichlet and Neumann cases;[1] this means that the concepts of trapping and nontrapping have no meaning under impedance boundary conditions. A key feature of the interior impedance problem when f in (1.1) equals zero is that the Cauchy data of the solution can be bounded in terms of g in (1.2) using Green's first identity. Since Green's integral representation gives the solution in the domain in terms of its Cauchy data on the boundary, k-explicit bounds on the norms of the integral operators can then be used to bound the solution in the domain by g. Similar ideas can be used to bound the solution when

[1] Indeed, adopting the convention for "outgoing" in the radiation condition (1.4) and letting $U(x,t)$ be the solution of the wave equation corresponding to $u(x;k)$, we find that, if $\eta = \pm k$, then the impedance boundary condition (1.2) corresponds to the boundary condition $\partial U/\partial n \pm \partial U/\partial t = \widetilde{g}$, under which energy is absorbed or emitted, respectively, by the boundary (assuming that the normal vector points outwards from the domain of propagation).

$f \neq 0$, with these arguments dating back to at least [20] (although these authors only considered the case when the domain is a square or cube). We use these ideas again here, with the main new ingredient being sharper bounds on the norms of the integral operators. These new bounds are obtained using the classic free resolvent estimates, and they result in sharper bounds on the solution. With some small modifications, this argument also yields a bound on the solution of the exterior impedance problem. Despite this BVP perhaps being less interesting than the others discussed so far, we also present the bound obtained on its solution.

Although the two parts of the paper (1 and 2 above) consider different problems, they are linked both by the methods they employ (with Vainberg's resolvent estimates and identities related to those of Morawetz playing key roles) and by the fact that the bounds in 1 and 2 together are then sufficient to obtain k-explicit bounds on the inverses of the standard second-kind boundary integral operators used to solve the exterior Dirichlet, Neumann, and impedance problems. (We illustrate this for the case of the Dirichlet problem in section 1.3.)

1.1. Statement of the main results. Let $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, Lipschitz open set with boundary $\Gamma := \partial\Omega_-$, such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $\gamma_\pm$ denote the trace operators from $\Omega_\pm$ to Γ, let $\partial_n^\pm$ denote the normal derivative trace operators, and let ∇_Γ denote the surface gradient operator on Γ. (For precise definitions of these operators, see section 2. Note that we will also call $\gamma_\pm u$ the Dirichlet traces of u and $\partial_n^\pm u$ the Neumann traces.) Let $B_R := \{\mathbf{x} : |\mathbf{x}| < R\}$.

This paper contains four theorems (Theorems 1.4, 1.5, 1.6, and 1.8). The first two concern the DtN and NtD maps for the Helmholtz equation in Ω_+ under geometric restrictions explained in the next three definitions.

Definition 1.1 (nontrapping). *We say that $\Omega_+ \subset \mathbb{R}^d$, $d = 2, 3$ is* nontrapping *if Γ is C^∞ and, given $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$, there exists a $T(R) < \infty$ such that all the billiard trajectories that start in $\Omega_+ \cap B_R$ at time zero leave $\Omega_+ \cap B_R$ by time $T(R)$.*

Definition 1.2 (nontrapping polygon). *If $\Omega_- \subset \mathbb{R}^2$ is a polygon, we say that it is a* nontrapping polygon *if* (i) *no three vertices are colinear and* (ii) *given $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$, there exists a $T(R) < \infty$ such that all the billiard trajectories that start in $\Omega_+ \cap B_R$ at time zero and miss the vertices leave $\Omega_+ \cap B_R$ by time $T(R)$. (For a more precise statement of* (ii)*, see* [8, section 5].*)*

Definition 1.3 (star-shaped). *Let $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, Lipschitz open set.*

(i) *We say that Ω_- is* star-shaped *if $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \Gamma$ for which $\mathbf{n}(\mathbf{x})$ is defined.*

(ii) *We say that Ω_- is* star-shaped with respect to a ball *if there exists a constant $c > 0$ such that $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq c$ for every $\mathbf{x} \in \Gamma$ for which $\mathbf{n}(\mathbf{x})$ is defined.*

Theorem 1.4 (bounds on the DtN map for the Helmholtz equation in exterior domains). *Let $d = 2$ or 3. Let $u \in H^1_{\mathrm{loc}}(\Omega_+)$ satisfy the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_+ \tag{1.3}$$

and the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r} - \mathrm{i}ku = o\left(\frac{1}{r^{(d-1)/2}}\right) \tag{1.4}$$

as $r := |\mathbf{x}| \to \infty$, uniformly in $\hat{\mathbf{x}} := \mathbf{x}/r$. If either *$\Omega_+$ is nontrapping (in the sense Definition* 1.1*)* or *Ω_- is a nontrapping polygon (in the sense of Definition* 1.2*)* or Ω_-

is Lipschitz and star-shaped (in the sense of Definition 1.3(i)*), then, given* $k_0 > 0$*,*

$$\left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} \lesssim k^{3/2} \left\|\gamma_+ u\right\|_{H^{1/2}(\Gamma)} \tag{1.5}$$

for all $k \geq k_0$*. Furthermore, if* $\gamma_+ u \in H^1(\Gamma)$*, then* $\partial_n^+ u \in L^2(\Gamma)$ *and, given* $k_0 > 0$*,*

$$\left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \lesssim k^{3/2} \left\|\gamma_+ u\right\|_{H^1(\Gamma)} \quad \text{and} \tag{1.6}$$

$$\left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \lesssim k \left(\left\|\nabla_\Gamma(\gamma_+ u)\right\|_{L^2(\Gamma)} + k \left\|\gamma_+ u\right\|_{L^2(\Gamma)} \right) \tag{1.7}$$

for all $k \geq k_0$*.*

How sharp are these bounds? By considering the specific examples of Γ the unit circle (in 2-d) and the unit sphere (in 3-d), we show that the bound (1.5) is at most $k^{1/2}$ away from being sharp (i.e., for the circle and sphere there exist solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition such that $\|\partial_n^+ u\|_{H^{-1/2}(\Gamma)} \gtrsim k \|\gamma_+ u\|_{H^{1/2}(\Gamma)}$), the bound (1.6) is at most $k^{1/2}$ away from being sharp, and the bound (1.7) is at most k away; see Lemma 3.10 for the details.

Theorem 1.5 (bounds on the NtD map for the Helmholtz equation in exterior domains). *Let* $d = 2$ *or* 3*. Let* $u \in H^1_{\mathrm{loc}}(\Omega_+)$ *satisfy the Helmholtz equation* (1.3) *and the Sommerfeld radiation condition* (1.4)*. If* either Ω_+ *is nontrapping (in the sense Definition* 1.1*)* or Ω_- *is a nontrapping polygon (in the sense of Definition* 1.2*), then, given* $k_0 > 0$*,*

$$\left\|\gamma_+ u\right\|_{H^{1/2}(\Gamma)} \lesssim k \left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} \tag{1.8}$$

for all $k \geq k_0$*. Furthermore, if* $\partial_n^+ u \in L^2(\Gamma)$*, then* $\gamma_+ u \in H^1(\Gamma)$ *and, given* $k_0 > 0$*,*

$$\left(\left\|\nabla_\Gamma(\gamma_+ u)\right\|_{L^2(\Gamma)} + k \left\|\gamma_+ u\right\|_{L^2(\Gamma)} \right) \lesssim k \left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \tag{1.9}$$

for all $k \geq k_0$*.*

By again considering the specific examples of Γ the unit circle and sphere, we show that the bound (1.8) is at most $k^{2/3}$ away from being sharp (i.e., for the circle and sphere there exist solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition such that $\|\gamma_+ u\|_{H^{1/2}(\Gamma)} \gtrsim k^{1/3} \|\partial_n^+ u\|_{H^{-1/2}(\Gamma)}$), and the bound (1.9) is at most $k^{2/3}$ away from being sharp.

The third theorem concerns the interior impedance problem for Ω_- a general bounded Lipschitz domain (where we use the word *domain* to mean a connected open set).

Theorem 1.6 (bounds on the solution to the interior impedance problem). *Let* Ω_- *be a bounded Lipschitz domain in* 2- *or* 3-*d. Given* $f \in L^2(\Omega_-)$*,* $g \in L^2(\Gamma)$*, and* $\eta \in \mathbb{R} \setminus \{0\}$*, let* $u \in H^1(\Omega_-)$ *be the solution to the interior impedance problem*

$$\Delta u + k^2 u = -f \quad \text{in } \Omega_- \quad \text{and} \quad \partial_n^- u - \mathrm{i}\eta\gamma_- u = g \quad \text{on } \Gamma. \tag{1.10}$$

Then, given $k_0 > 0$*,*

$$\left\|\nabla u\right\|_{L^2(\Omega_-)} + k \left\|u\right\|_{L^2(\Omega_-)} \lesssim k^{1/2} \left(1 + \frac{k}{|\eta|}\right) \left[\left\|g\right\|_{L^2(\Gamma)} + k^{1/2} \left(1 + \frac{|\eta|}{k}\right) \left\|f\right\|_{L^2(\Omega_-)} \right] \tag{1.11}$$

for all $k \geq k_0$ *(where the omitted constant is independent of both* k *and* η*). In particular, if* $|\eta| \sim k$*, then*

$$\left\|\nabla u\right\|_{L^2(\Omega_-)} + k \left\|u\right\|_{L^2(\Omega_-)} \lesssim k^{1/2} \left\|g\right\|_{L^2(\Gamma)} + k \left\|f\right\|_{L^2(\Omega_-)}. \tag{1.12}$$

Furthermore, if Γ *is piecewise smooth, then the* $k^{1/2}$ *at the front of the left-hand side of* (1.11) *can be replaced by* $k^{1/4}$*, and thus if* $|\eta| \sim k$*, then*

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim k^{1/4}\,\|g\|_{L^2(\Gamma)} + k^{3/4}\,\|f\|_{L^2(\Omega_-)}\,. \tag{1.13}$$

In Lemma 4.10 we investigate the sharpness of (1.12) and (1.13). (For simplicity we restrict attention to the case $|\eta| = k$, but the methods we use are applicable for general η.) We show that the factor in front of $\|g\|_{L^2(\Gamma)}$ in (1.12) is at most k away from being sharp, and the factor in front of $\|f\|_{L^2(\Omega_-)}$ in (1.12) is k away from being sharp. Analogously, the factors in front of $\|g\|_{L^2(\Gamma)}$ and $\|f\|_{L^2(\Omega_-)}$ in (1.13) are both $k^{3/4}$ away from being sharp.

Theorem 1.6 can be used to prove a bound on the solution of the interior impedance problem with minimal smoothness requirements on the data, and this gives a bound on the inf-sup constant of the corresponding variational formulation.

Corollary 1.7 (corollary to Theorem 1.6). *Given* $k_0 > 0$*, the solution of the interior impedance problem with* $f \in (H^1(\Omega_-))'$*,* $g \in H^{-1/2}(\Gamma)$*, satisfies*

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim k^2\left(1+\frac{k}{|\eta|}\right)\left(1+\frac{|\eta|}{k}\right)\left[\|g\|_{H^{-1/2}(\Gamma)} + \|f\|_{(H^1(\Omega_-))'}\right] \tag{1.14}$$

for all $k \geq k_0$*. Therefore, in the case* $|\eta| \sim k$*, the sesquilinear form of the variational formulation of the interior impedance problem,* $a(\cdot,\cdot)$ *defined by* (4.2) *below, satisfies*

$$\inf_{0\neq u\in H^1(\Omega_-)}\ \sup_{0\neq v\in H^1(\Omega_-)} \frac{|a(u,v)|}{\|u\|_{1,k,\Omega_-}\,\|v\|_{1,k,\Omega_-}} \gtrsim \frac{1}{k^2}, \tag{1.15}$$

where $\|u\|_{1,k,\Omega_-} := \|\nabla u\|_{L^2(\Omega_-)} + k\|u\|_{L^2(\Omega_-)}$*. If* Γ *is piecewise smooth, then the factor of* k^2 *both on the right-hand side of* (1.14) *and in the denominator of the right-hand side of* (1.15) *can be changed to* $k^{7/4}$*.*

The final theorem concerns the exterior impedance problem for Ω_- a general Lipschitz domain.

Theorem 1.8 (bounds on the solution to the exterior impedance problem). *Let* Ω_- *be a bounded Lipschitz domain in 2- or 3-d. Given* $f \in L^2(\Omega_+)$ *with compact support,* $g \in L^2(\Gamma)$*, and* $\eta > 0$*, let* $u \in H^1_{\mathrm{loc}}(\Omega_+)$ *be the solution to the exterior impedance problem*

$$\Delta u + k^2 u = -f \quad \text{in } \Omega_+ \quad \text{and} \quad \partial_n^+ u + \mathrm{i}\eta\gamma_+ u = g \quad \text{on } \Gamma, \tag{1.16}$$

satisfying the Sommerfeld radiation condition (1.4)*. Then, for any* $R > \sup_{\mathbf{x}\in\Omega_-}|\mathbf{x}|$*, the bound* (1.11) *holds with the left-hand side replaced by*

$$\|\nabla u\|_{L^2(\Omega_R)} + k\,\|u\|_{L^2(\Omega_R)}\,,$$

where $\Omega_R := \Omega_+ \cap \{|\mathbf{x}| < R\}$*, and with* $\|f\|_{L^2(\Omega_-)}$ *replaced by* $\|f\|_{L^2(\Omega_+)}$*. Furthermore, if* Γ *is piecewise smooth, then the factor of* $k^{1/2}$ *on the right-hand side of this bound can be replaced by* $k^{1/4}$*.*

Recall that, while the interior impedance problem has a unique solution for all $\eta \in \mathbb{R}\setminus\{0\}$, the exterior impedance problem needs η in the boundary condition in (1.16) to be greater than zero for the solution to be unique (and so this restriction is in the statement of the theorem); see, e.g., [13, Theorem 3.37], [11, Lemma 2.8].

Regarding sharpness. As in the case of the interior problem, the argument in the proof of Lemma 4.10 shows that the factor in front of $\|g\|_{L^2(\Gamma)}$ in the analogue of

(1.12) is at most k away from being sharp, and the factor in front of $\|f\|_{L^2(\Omega_+)}$ is k away from being sharp. A corollary analogous to Corollary 1.7 holds for the exterior impedance problem, but we omit the details.

1.2. Comparison of the main results to similar existing results.

Bounds on the DtN and NtD maps (Theorems 1.4 *and* 1.5*).* In this discussion we omit results about the high-frequency asymptotics of the solution of the Helmholtz equation in Ω_+. There has been vast amounts of research on constructing these asymptotics and justifying them rigorously; for an introduction to this work, see, e.g., [4], [5], [11, section 3], and the references therein.

Instead, we focus on results that specifically bound either the DtN or the NtD map (such as Theorems 1.4 and 1.5). To the author's knowledge, there exist four such results. The first of these was obtained by Morawetz and Ludwig in [45]. They proved that if Ω_- is smooth and star-shaped with respect to a ball (in the sense of Definition 1.3(ii)), then, given $k_0 > 0$,

$$\left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \lesssim \|\nabla_\Gamma(\gamma_+ u)\|_{L^2(\Gamma)} + k\,\|\gamma_+ u\|_{L^2(\Gamma)} \tag{1.17}$$

for all $k \geq k_0$. This result was obtained using the identity for solutions of the Helmholtz equation that arises by multiplying the PDE by $\overline{\mathcal{M}u}$, where

$$\mathcal{M}u = \mathbf{x}\cdot\nabla u - \mathrm{i}kr + \frac{(d-1)}{2}u. \tag{1.18}$$

(For a discussion of why this is possible, see the review [11, section 5.3.1].) With some additional technical work this method can be applied when Ω_- is a Lipschitz, star-shaped domain, and thus the bound (1.17) also holds in this case. (See Remark 3.8 for more details.)

The second result is a bound on the NtD map obtained by Babich in [3]. Babich proved that if Ω_- is a smooth, convex, two-dimensional domain with strictly positive curvature, then

$$\|\gamma_+ u\|_{L^\infty(\Gamma)} \lesssim \frac{1}{k^{1/2}}\left\|\partial_n^+ u\right\|_{L^\infty(\Gamma)} \tag{1.19}$$

for all $k > 0$. This result was obtained using a method introduced by Ursell in [56] (and then also used in, e.g., [2], [21], [57]). The method approximates the Neumann Green's function for Ω_+ with source at $\mathbf{x}_0 \in \Gamma$ with the Neumann Green's function for the exterior of the osculating circle at $\mathbf{x}_0$. This approximate Green's function is then used to formulate an integral equation for the solution of the Neumann problem in Ω_+. Since the Green's function for the circle is known explicitly, the bound (1.19) can then be obtained from the integral equation.

The third and fourth results are the following bounds on the DtN and NtD map for nontrapping domains (in the sense of Definition 1.1) obtained by Lakshtanov and Vainberg in [28, Theorem 1]: given $k_0 > 0$,

$$\left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} \lesssim k^3\,\|\gamma_+ u\|_{H^{1/2}(\Gamma)} \quad \text{and} \quad \|\gamma_+ u\|_{H^{1/2}(\Gamma)} \lesssim k\left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} \tag{1.20}$$

for all $k \geq k_0$. As discussed above, these bounds were obtained using the resolvent estimate for this class of domains obtained by Vainberg in [58] (and we use essentially the same method in section 3 to prove the bounds in Theorems 1.4 and 1.5).

We now compare these four previous results with the bounds in Theorems 1.4 and 1.5. The Morawetz–Ludwig DtN bound (1.17) is sharper in its k-dependence than the bound on the DtN map (1.7), although (1.7) holds for a wider range of geometries than (1.17). Note that the specific examples of the circle and sphere, analyzed in Lemma 3.12, show that the Morawetz–Ludwig bound (1.17) is sharp in its k-dependence.

The DtN bound in the trace spaces (1.5) is sharper than that of Lakshtanov and Vainberg in (1.20), but the NtD bound in the trace spaces (1.8) is the same as that of Lakshtanov and Vainberg in (1.20) (although both (1.5) and (1.8) hold for a wider range of geometries than the bounds in (1.20)). We note that the investigation in [28] was not focused on obtaining the best possible bounds on the DtN and NtD maps, since the powers of k in the bounds (1.20) were sufficient for proving the main results of [28] (sharp bounds on the total cross-sections of scattered waves when either Ω_+ is nontrapping or Ω_- is a general Lipschitz domain).

The Babich bound (1.19) cannot immediately be compared to the NtD bounds (1.8) and (1.9), since the spaces in which the bounds are proved are different. Nevertheless, the particular examples of the circle and sphere show that the Babich bound is at most $k^{1/6}$ away from being sharp (see Remark 3.13), and the NtD bounds (1.8) and (1.9) are both at most $k^{2/3}$ away from being sharp.

Before leaving this discussion on bounds on the DtN and NtD maps, we note that if the domain is trapping, then one cannot expect bounds such as those above to hold. For example, if Ω_+ is a two-dimensional domain with an elliptical cavity, in the sense that Ω_+ contains the ellipse $\{(x_1,x_2) : (x_1/a_1)^2 + (x_2/a_2)^2 < 1\}$ with $a_1 > a_2 > 0$ and Γ coincides with the boundary of the ellipse in neighborhoods of $(0, \pm a_2)$, then there exist wavenumbers $0 < k_1 < k_2 < \cdots$ with $k_m \to \infty$ as $m \to \infty$, corresponding solutions of the Helmholtz equation that satisfy the Sommerfeld radiation condition u_m, and a constant $\gamma > 0$ such that

$$\left\|\partial_n^+ u_m\right\|_{L^2(\Gamma)} \gtrsim \mathrm{e}^{\gamma k_m}\left(\left\|\nabla_\Gamma(\gamma_+ u_m)\right\|_{L^2(\Gamma)} + k_m \left\|\gamma_+ u_m\right\|_{L^2(\Gamma)}\right)$$

for all $m \geq 1$. (This can be proved using techniques similar to those in [9, Theorem 2.8]; see also the discussion in [11, section 5.6.1].)

Bounds on the interior and exterior impedance problems (Theorems 1.6 *and* 1.8*).* For simplicity we consider the case that $|\eta| = k$. Some of the previous results that we now discuss only considered this case, although the methods used to prove these results also work for general η.

If Ω_- is a two- or three-dimensional Lipschitz domain that is star-shaped with respect to a ball (in the sense of Definition 1.3(ii)) then the identity resulting from the multiplier

$$\mathcal{M}u = \mathbf{x}\cdot\nabla u + \frac{(d-1)}{2}u \tag{1.21}$$

can be used to prove that, given $k_0 > 0$,

$$\left\|\nabla u\right\|_{L^2(\Omega_-)} + k\left\|u\right\|_{L^2(\Omega_-)} \lesssim \left\|g\right\|_{L^2(\Gamma)} + \left\|f\right\|_{L^2(\Omega_-)} \tag{1.22}$$

for all $k \geq k_0$. This was done when Γ is piecewise smooth in 2-d by Melenk [33, Proposition 8.1.4] and in 3-d by Cummings and Feng [15, Theorem 1]. The arguments outlined in Remark 3.8 can then be used to establish the bound when Γ is Lipschitz (similar to the situation with the Morawetz–Ludwig DtN bound discussed above). Lemma 4.10 shows that, at least when $g = 0$, the bound (1.22) is sharp in its k-dependence.

The argument involving Green's integral representation and k-explicit bounds on integral operators that we discussed above was used by Feng and Sheen to prove that if Ω_- is square or cube and $g = 0$, then, given $k_0 > 0$,

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim k^2\,\|f\|_{L^2(\Omega_-)} \tag{1.23}$$

for all $k \geq k_0$ [20, Theorems 3.6 and 4.7]; the same argument can be used to establish the bound when Ω_- is a general Lipschitz domain. This argument was used independently by Esterhazy and Melenk to prove that, with Ω_- a general two- or three-dimensional Lipschitz domain, given $k_0 > 0$,

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim k^2\,\|g\|_{L^2(\Gamma)} + k^{5/2}\,\|f\|_{L^2(\Omega_-)} \tag{1.24}$$

for all $k \geq k_0$ [18, Theorem 2.4].

Looking at these previous results, we see that the bound (1.12) of Theorem 1.6 is the sharpest yet obtained in the case that Ω_- is a general Lipschitz domain, but the k-dependence is still worse than that in the bound (1.22) for domains that are star-shaped with respect to a ball. The bound (1.13) improves the k-dependence in the case when Γ is piecewise smooth, but this improved dependence is still not as good as that in the star-shaped case.

To the author's knowledge, there are currently no bounds for the exterior impedance problem stated in the literature (although, as we see in this paper, the method used to prove the interior bounds (1.11), (1.23), and (1.24) can easily be adapted to prove exterior bounds).

1.3. Conditioning of boundary integral operators. As discussed above, one application of the bounds of Theorems 1.4, 1.5, 1.6, and 1.8 is in proving bounds on the inverses of boundary integral operators (which can then be used in conjunction with bounds on the norms of these operators to prove bounds on their condition numbers). We illustrate this for the standard second-kind integral operator used to solve the exterior Dirichlet problem.

When u is the solution to the exterior Dirichlet problem for the Helmholtz equation, the Neumann trace of u, $\partial_n^+ u$, satisfies the integral equation

$$A'_{k,\eta}(\partial_n^+ u) = f \tag{1.25}$$

on Γ, where the integral operator $A'_{k,\eta}$ is the so-called combined-potential or combined-field integral operator (defined by (1.30) below) and f is given in terms of the known Dirichlet data $\gamma_+ u$.

We now briefly derive the integral equation (1.25); for simplicity we do not consider the general exterior Dirichlet problem, only the *sound-soft scattering problem* (i.e., the problem in which the Dirichlet data is the restriction of, e.g., a plane wave to Γ). The reason we do this is that the right-hand side f of (1.25) takes a particularly simple form in this case; for the details of the general case see [11, equations (2.68) and (2.69)]).

Definition 1.9 (sound-soft scattering problem). *Given $k > 0$ and an incident plane wave $u^I(\mathbf{x}) = \exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $|\widehat{\mathbf{a}}| = 1$, find $u^S \in C^2(\Omega_+) \cap H^1_{\mathrm{loc}}(\Omega_+)$ such that the total field $u := u^I + u^S$ satisfies*

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_+, \qquad \gamma_+ u = 0 \quad \text{on } \Gamma,$$

and u^S satisfies the Sommerfeld radiation condition (1.4) *(i.e.,* (1.4) *holds with u replaced by u^S).*

Using (i) the fact that u^I is a solution of the Helmholtz equation in Ω_-, and (ii) Green's integral representation for u^S, one can show that

$$u(\mathbf{x}) = u^I(\mathbf{x}) - \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\partial_n^+ u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Omega_+ \tag{1.26}$$

(see, e.g., [11, Theorem 2.43]), where $\Phi_k(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4}H_0^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big),\ d=2, \qquad \Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},\ d=3. \tag{1.27}$$

Taking the Dirichlet and Neumann traces of (1.26) on Γ and using the jump relations for the single-layer potential (given in (5.1) below), one obtains two integral equations for the unknown Neumann boundary value $\partial_n^+ u$:

$$S_k\partial_n^+ u = \gamma_+ u^I, \qquad \left(\frac{1}{2}I + D_k'\right)\partial_n^+ u = \partial_n^+ u^I, \tag{1.28}$$

where the integral operators S_k and D_k', the single-layer operator and the adjoint-double-layer operator, respectively, are defined for $\psi \in L^2(\Gamma)$ by

(1.29)
$$S_k\psi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad D_k'\psi(\mathbf{x}) := \int_\Gamma \frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma.$$

(When Γ is Lipschitz, the integral defining D_k' is understood as a Cauchy principal value integral; see, e.g., [11, section 2.3].)

Both integral equations in (1.28) fail to be uniquely solvable for certain values of k. (For the first equation in (1.28) these are the k such that k^2 is a Dirichlet eigenvalue of the Laplacian in Ω_-, and for the second equation in (1.28) these are the k such that k^2 is a Neumann eigenvalue.) The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation (1.25), where

$$A_{k,\eta}' := \frac{1}{2}I + D_k' - \mathrm{i}\eta S_k, \tag{1.30}$$

the so-called coupling parameter $\eta \in \mathbb{R}\setminus\{0\}$,[2] and

$$f(\mathbf{x}) := \partial_n^+ u^I(\mathbf{x}) - \mathrm{i}\eta\gamma_+ u^I(\mathbf{x}), \quad \mathbf{x}\in\Gamma. \tag{1.31}$$

The integral equation (1.25) is usually considered as an equation in the space $L^2(\Gamma)$, since $A_{k,\eta}'$ is a bounded and invertible operator on $L^2(\Gamma)$ (when $\eta\in\mathbb{R}\setminus\{0\}$) [11, Theorem 2.27], and both $\partial_n^+ u$ and $f\in L^2(\Gamma)$ in the case of plane-wave or point-source scattering [11, Theorems 2.12 and 2.46].

Although integral equations such as (1.25) have long been used to solve scattering problems, until recently little has been known about how quantities of interest

[2] Although denoting the coupling parameter η might appear to be a notational clash with the η in the impedance boundary condition (1.2), the adjoint of the integral operator $A_{k,\eta}'$ can be used to solve the interior impedance problem, and in this case the coupling parameter equals the η in the impedance boundary condition; see [11, Theorem 2.30]. This relationship between the coupling parameter and the η in the impedance boundary condition can also be seen in Lemma 1.10 below.

(such as the norms of the operators) depend on k and η. It turns out that bounds on the norm of $A'_{k,\eta}$ that are explicit in k and η can be obtained using standard techniques for bounding norms of integral operators [11, section 5.5], [53, section 1.2, section 1.4]. However, to obtain bounds on $(A'_{k,\eta})^{-1}$ that are explicit in k and η, one must use results about the exterior DtN map and the interior impedance-to-Dirichlet map. Indeed, the following lemma is implicit in [12, Proof of Lemma 4.5] and [11, Theorem 2.33], and proved explicitly in section 5.

LEMMA 1.10 (bounding the inverse of the combined potential operator). *Let u_+ satisfy $\Delta u_+ + k^2 u_+ = 0$ in Ω_+, the Sommerfeld radiation condition* (1.4)*, and let $\gamma_+ u_+ \in H^1(\Gamma)$. Let u_- be the solution of the interior impedance problem* (1.10) *with $f = 0$, $g \in L^2(\Gamma)$, and $\eta \in \mathbb{R} \setminus \{0\}$. If α, β, and δ are such that, given $k_0 > 0$,*

$$\left\|\partial_n^+ u_+\right\|_{L^2(\Gamma)} \lesssim \alpha \left\|\nabla_\Gamma(\gamma_+ u_+)\right\|_{L^2(\Gamma)} + \beta k \left\|\gamma_+ u_+\right\|_{L^2(\Gamma)} \tag{1.32}$$

and

$$\left\|\nabla_\Gamma(\gamma_- u_-)\right\|_{L^2(\Gamma)} \lesssim \delta \left\|g\right\|_{L^2(\Gamma)} \tag{1.33}$$

for all $k \geq k_0$, then

$$\left\|(A'_{k,\eta})^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \left(1 + \alpha\delta + \beta\frac{k}{|\eta|}\right) \tag{1.34}$$

for all $k \geq k_0$ and $\eta \in \mathbb{R} \setminus \{0\}$.

This lemma implies that if one can bound both the exterior DtN map and the interior impedance-to-Dirichlet map, then one can bound $(A'_{k,\eta})^{-1}$. Similarly, if one can bound the exterior NtD map and the interior impedance-to-Dirichlet map, then one can bound the inverse of the standard second-kind boundary integral operator used to solve the exterior Neumann problem, and if one can bound both the exterior and interior impedance-to-Dirichlet maps, then one can bound the inverse of the standard second-kind boundary integral operator used to solve the exterior impedance problem; see [11, Theorem 2.33].

If Ω_- is a two- or three-dimensional Lipschitz domain that is star-shaped with respect to a ball (in the sense of Definition 1.3(ii)), then the Morawetz–Ludwig DtN bound (1.17) implies that (1.32) holds with α and $\beta \sim 1$. Furthermore, the bound on the interior impedance problem (1.22) for this class of domains can be used to show that (1.33) holds with $\delta \sim 1 + k/|\eta|$ (see Remark 4.8). Lemma 1.10 then implies that

$$\left\|(A'_{k,\eta})^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + \frac{k}{|\eta|} \tag{1.35}$$

when Ω_- is a two- or three-dimensional Lipschitz domain that is star-shaped with respect to a ball; this result was first proved in [12, Theorem 4.3].

Using the bounds of Theorems 1.4 and 1.6 in Lemma 1.10, we obtain the following theorem.

THEOREM 1.11 (bound on $(A'_{k,\eta})^{-1}$ for smooth nontrapping domains and nontrapping polygons). *If* either *$\Omega_+ \subset \mathbb{R}^d$, $d = 2, 3$, is nontrapping (in the sense of Definition* 1.1*) or Ω_- is a nontrapping polygon (in the sense of Definition* 1.2*), then, given $k_0 > 0$,*

$$\left\|(A'_{k,\eta})^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{5/4}\left(1 + \frac{k^{3/4}}{|\eta|}\right) \tag{1.36}$$

for all $k \geq k_0$ and $\eta \in \mathbb{R} \setminus \{0\}$.

Proof. The bound (1.7) implies that (1.32) holds with α and $\beta \sim k$. Corollary 4.7 shows that the analogue of the bound (1.11) when Γ is piecewise smooth implies that (1.33) holds with $\delta \sim k^{1/4}(1 + k^{3/4}/|\eta|)$. The bound (1.36) then follows from Lemma 1.10. ∎

The numerical experiments in [10] indicate that the bound (1.36) is not sharp in its k-dependence, since they show that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ is bounded independently of k when Ω_- is a particular nontrapping and non-star-shaped polygon and $\eta = k$ (see [10, Figure 5.9]). This lack of sharpness is to be expected, since both the bounds used to obtain (1.36), namely, (1.7) and (1.11), are not sharp.

Despite its lack of sharpness, the bound (1.36) is sufficient for the following numerical analysis application: Löhndorf and Melenk have recently performed a k-explicit convergence analysis of the Galerkin method applied to the integral equation (1.25) using piecewise-polynomial subspaces (the so-called hp-boundary-element method) [31], [34]. An underlying assumption in this analysis is that, when $|\eta| \sim k$, $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^a$ for some $a > 0$. This assumption was known to hold for Lipschitz star-shaped domains via the bound (1.35), and Theorem 1.11 now establishes that this assumption holds for nontrapping domains in 2- or 3-d and for nontrapping polygons.

1.4. Outline of paper. In section 2 we establish some notation and collect some basic results that are used throughout the paper. In section 3 we prove Theorems 1.4 and 1.5 (the bounds on the DtN and NtD maps). In section 4 we prove Theorems 1.6 and 1.8 (the bounds on the interior and exterior impedance problems). In section 5 we prove Lemma 1.10.

2. Notation and basic results. We use the notation $a \lesssim b$ to mean $a \leq Cb$ for some constant C that is independent of k, η, and any other parameters of interest. (Usually these will be explicitly stated.) $a \gtrsim b$ means $b \lesssim a$. If $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$.

Let $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, Lipschitz open set with boundary $\Gamma := \partial\Omega_-$, such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $\mathbf{n}$ denote the outward-pointing, unit, normal vector to Ω_-. Let $B_R := \{\mathbf{x} : |\mathbf{x}| < R\}$, let $\Gamma_R := \{\mathbf{x} : |\mathbf{x}| = R\}$, and let $\Omega_R := \Omega_+ \cap B_R$.

We denote the interior and exterior traces by $\gamma_\mp$, so that, for $1/2 < s < 3/2$, $\gamma_- : H^s(\Omega_-) \to H^{s-1/2}(\Gamma)$ and $\gamma_+ : H^s_{\rm loc}(\Omega_+) \to H^{s-1/2}(\Gamma)$. We have the bound

$$\|\gamma_- u\|_{H^{s-1/2}(\Gamma)} \lesssim \|u\|_{H^s(\Omega_-)} \quad \text{for } 1/2 < s < 3/2 \tag{2.1}$$

[14, Lemma 3.6], [32, Theorem 3.38], and the multiplicative trace inequality

$$\|\gamma_- u\|^2_{L^2(\Gamma)} \lesssim \|u\|_{L^2(\Omega_-)} \|u\|_{H^1(\Omega_-)} \tag{2.2}$$

[22, Theorem 1.5.1.10, last formula on p. 41]. If $\chi \in C^\infty_{\rm comp}(\mathbb{R}^d)$ and $\chi = 1$ in a neighborhood of Γ, then $\gamma_\pm(\chi u) = \gamma_\pm(u)$ for all $u \in H^s_{\rm loc}(\Omega_+)$ with $1/2 < s < 3/2$ [51, Remark 2.6.10]. Therefore, if $u \in H^s_{\rm loc}(\Omega_+)$ and $1/2 < s < 3/2$, then

$$\|\gamma_+ u\|_{H^{s-1/2}(\Gamma)} \lesssim \|\chi u\|_{H^s(\Omega_+)}, \tag{2.3}$$

and if $u \in H^1_{\rm loc}(\Omega_+)$, then

$$\|\gamma_+ u\|^2_{L^2(\Gamma)} \lesssim \|\chi u\|_{L^2(\Omega_+)} \|\chi u\|_{H^1(\Omega_+)}. \tag{2.4}$$

Denote the surface gradient on Γ by ∇_Γ; see, e.g., [11, equation (A.14)] for the definition of this operator in terms of a parametrization of the boundary. Recall that ∇_Γ is a bounded operator from $H^1(\Gamma)$ to $(L^2(\Gamma))^d$ and, furthermore, if $f \in H^1(\Gamma)$, then

$$\|f\|_{H^1(\Gamma)} \sim \|\nabla_\Gamma f\|_{L^2(\Gamma)} + \|f\|_{L^2(\Gamma)}. \tag{2.5}$$

The space $H^1(\Omega_-, \Delta)$ is defined to be equal to $\{u : u \in H^1(\Omega_-), \Delta u \in L^2(\Omega_-)\}$ and $H^1_{\mathrm{loc}}(\Omega_+, \Delta) := \{u : u \in H^1_{\mathrm{loc}}(\Omega_+), \Delta u \in L^2_{\mathrm{loc}}(\Omega_+)\}$.

Let $\partial_n^\pm$ denote the normal-derivative traces on $\Omega_\pm$ (recalling our convention that the normal vector points out of Ω_-). Recall that if $u \in H^2(\Omega_-)$, then $\partial_n^- u := \mathbf{n} \cdot \gamma_-(\nabla u)$, and for $u \in H^1(\Omega_-, \Delta)$, $\partial_n^- u$ is defined so that Green's first identity holds (see, e.g., [11, equation (A.29)]).

Lemma 2.1 (Green's first identity). *With D a Lipschitz domain, if $u \in H^1(D, \Delta)$ and $v \in H^1(D)$, then*

$$\langle \partial_n u, \gamma v\rangle_{\partial D} = \int_D \left(\nabla u \cdot \overline{\nabla v} + \overline{v}\Delta u\right) \mathrm{d}\mathbf{x}, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle_{\partial D}$ denotes the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.

Whenever we say that u satisfies $\Delta u + k^2 u = -f$ (for a given f), we always mean that this equation is satisfied in a distributional sense. Note that interior regularity of the Laplacian then implies that u is C^∞ outside the support of f and away from the boundary (see, e.g., [32, Theorem 4.16], [19, section 6.3.1]). Therefore, if the PDE is posed in Ω_+ and f has compact support, the Sommerfeld radiation condition (1.4) can legitimately be imposed.

Later in the paper, we consider the modified Helmholtz equation $\Delta v - \lambda^2 v = 0$ in Ω_+ for $\lambda > 0$ with the condition that v is bounded at infinity. Interior regularity of the Laplacian, separation of variables, and asymptotics of modified Bessel functions then imply that $v(\mathbf{x}) \sim \exp(-\lambda r) r^{-(d-1)/2}$ as $r := |\mathbf{x}| \to \infty$, and thus both v and ∇v are in $L^2(\Omega_+)$.

We repeatedly use the inequality

$$\sqrt{\sum_{j=1}^n a_j^2} \le \sum_{j=1}^n a_j \lesssim \sqrt{\sum_{j=1}^n a_j^2} \quad \text{for } a_j \ge 0,$$

as well as the inequality

$$2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon} \quad \text{for } a, b, \text{ and } \varepsilon > 0. \tag{2.7}$$

(Following [19] we refer to (2.7) as the Cauchy inequality.)

We show in the next lemma that the H^1-norm of a solution of the Helmholtz equation in Ω_+ can be bounded by the L^2-norms of the solution and the data. Variants of this lemma can be found in [44, Lemma 1] and [9, proof of Theorem 2.8].

Lemma 2.2 (bounding the H^1 norm via the L^2 norm and the data). *Given $k > 0$ and $f \in L^2(\Omega_+)$ with compact support, let $u \in H^1_{\mathrm{loc}}(\Omega_+, \Delta)$ be a solution of the Helmholtz equation $\Delta u + k^2 u = -f$ in Ω_+.*

(a) *If* either $\gamma_+ u = 0$ or $\partial_n^+ u = 0$, *then for any $R > \sup_{\mathbf{x} \in \Omega_-} |\mathbf{x}|$, given $k_0 > 0$,*

$$\|\nabla u\|_{L^2(\Omega_R)} \lesssim k\|u\|_{L^2(\Omega_{R+1})} + k^{-1}\|f\|_{L^2(\Omega_+)} \tag{2.8}$$

for all $k \ge k_0$.

(b) *If* $\partial_n^+ u + \mathrm{i}\eta\gamma_+ u = g$ *on* Γ*, where* $g \in L^2(\Gamma)$ *and* $\eta \in \mathbb{R}$*, then, for any* $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$*, given* $k_0 > 0$*,*

$$\|\nabla u\|_{L^2(\Omega_R)} \lesssim k\,\|u\|_{L^2(\Omega_{R+1})} + k^{-1}\,\|f\|_{L^2(\Omega_+)} + k^{-1/2}\,\|g\|_{L^2(\Gamma)} \tag{2.9}$$

for all $k \geq k_0$*.*

Proof.

(a) Let $F \in C^1[0, R+1]$ be such that (i) $F = 1$ on $[0, R]$, (ii) $0 \leq F(s) \leq 1$ for $s \in [R, R+1]$, (iii) $F(R+1) = 0$, and (iv) there exists an $M > 0$ such that $(F'(s))^2/F(s) \leq M$ for $s \in [0, R+1]$. (This last condition can be achieved by requiring that F vanishes quadratically at $R+1$.) Let $\chi(\mathbf{x}) := F(|\mathbf{x}|)$. Then $\chi u \in H^1(\Omega_{R+1})$ with $\gamma(\chi u) = 0$ on Γ_{R+1} and $\gamma_+(\chi u) = \gamma_+ u$. Applying Green's identity (2.6) in Ω_{R+1} with $v = \chi u$, we obtain

$$\int_{\Omega_{R+1}} \chi|\nabla u|^2 \mathrm{d}\mathbf{x} = \int_{\Omega_{R+1}} \left(k^2\chi|u|^2 - \overline{u}\nabla u \cdot \nabla\chi + \chi\overline{u}f\right) \mathrm{d}\mathbf{x}, \tag{2.10}$$

where we have used the facts that both $\langle \partial_n^+ u, \gamma_+ u\rangle_\Gamma$ and $\langle \partial_n u, \gamma(\chi u)\rangle_{\Gamma_{R+1}}$ are zero.

Using the Cauchy inequality (2.7), we have

$$\begin{aligned} \left|\int_{\Omega_{R+1}} \overline{u}\nabla u \cdot \nabla\chi \,\mathrm{d}\mathbf{x}\right| &\leq \int_{\Omega_{R+1}} \chi^{1/2}|u||\nabla u|\frac{|\nabla\chi|}{\chi^{1/2}}\,\mathrm{d}\mathbf{x} \\ &\leq \frac{\varepsilon}{2}\int_{\Omega_{R+1}} \chi|\nabla u|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2\varepsilon}\int_{\Omega_{R+1}} |u|^2\frac{|\nabla\chi|^2}{\chi}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{2.11}$$

and

$$\left|\int_{\Omega_{R+1}} \chi\overline{u}f\,\mathrm{d}\mathbf{x}\right| \leq \frac{\delta}{2}\int_{\Omega_{R+1}} \chi|u|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2\delta}\int_{\Omega_{R+1}} \chi|f|^2\,\mathrm{d}\mathbf{x} \tag{2.12}$$

for any ε and $\delta > 0$. Choosing $\varepsilon = 1$ and $\delta = k^2$ and using (2.11) and (2.12) in (2.10), we obtain

$$\begin{aligned} \frac{1}{2}\int_{\Omega_{R+1}} \chi|\nabla u|^2\,\mathrm{d}\mathbf{x} \leq \frac{3k^2}{2}\int_{\Omega_{R+1}} \chi|u|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\Omega_{R+1}} |u|^2\frac{|\nabla\chi|^2}{\chi}\,\mathrm{d}\mathbf{x} \\ + \frac{1}{2k^2}\int_{\Omega_{R+1}} \chi|f|^2\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.13}$$

Since $\chi \geq 0$ on Ω_{R+1} and $\chi = 1$ on Ω_R, the left-hand side of (2.13) is $\geq \|\nabla u\|^2_{L^2(\Omega_R)}/2$. Condition (iv) above on the function F implies that $|\nabla\chi|^2/\chi$ is bounded on Ω_{R+1}. Using this fact in the right-hand side of (2.13), along with the fact that $\chi \leq 1$ on Ω_{R+1}, we obtain the result (2.8).

(b) This is very similar to the proof of (a), with the only differences being (i) one takes the real part of the analogue of (2.10) (to eliminate a term involving $\|\gamma_+ u\|^2_{L^2(\Gamma)}$), (ii) at the end one uses the multiplicative trace (2.4) and Cauchy (2.7) inequalities to obtain

$$k\,\|\gamma_+ u\|^2_{L^2(\Gamma)} \lesssim \left(\varepsilon + \frac{k^2}{\varepsilon}\right)\|u\|^2_{L^2(\Omega_R)} + \varepsilon\,\|\nabla u\|^2_{L^2(\Omega_R)},$$

and then one must choose ε sufficiently small when using this inequality to obtain the result (2.9). $\square$

We now prove an interpolation result that allows us to "move" bounds on the DtN and NtD maps between Sobolev spaces. To state this result, we denote the DtN map in Ω_+ by P^+_{DtN} and the NtD map by P^+_{NtD} (following the notation in [11, section 2.7]). P^+_{DtN} is defined as a map from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ by standard results about the solvability of the exterior Dirichlet problem and the definition of the normal derivative. A result of Nečas [47, sections 5.1.2 and 5.2.1], [32, Theorem 4.24] (discussed in more detail in section 3.3 below) implies that P^+_{DtN} can be extended to a map from $H^1(\Gamma)$ to $L^2(\Gamma)$, and then a representation of P^+_{DtN} in terms of boundary-integral operators means that P^+_{DtN} can be extended to a map from $H^{s+1/2}(\Gamma)$ to $H^{s-1/2}(\Gamma)$ for $|s| \leq 1/2$ (see [11, Theorem 2.31]). Analogous arguments show that P^+_{NtD} can be extended to a map from $H^{s-1/2}(\Gamma)$ to $H^{s+1/2}(\Gamma)$ for $|s| \leq 1/2$.

LEMMA 2.3. *With Ω_+, P^+_{DtN}, and P^+_{NtD} defined above,*

$$\left\|P^+_{DtN}\right\|_{L^2(\Gamma)\to H^{-1}(\Gamma)} = \left\|P^+_{DtN}\right\|_{H^1(\Gamma)\to L^2(\Gamma)}, \tag{2.14}$$

$$\left\|P^+_{DtN}\right\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \leq \left\|P^+_{DtN}\right\|_{H^1(\Gamma)\to L^2(\Gamma)}, \tag{2.15}$$

and analogously,

$$\left\|P^+_{NtD}\right\|_{H^{-1}(\Gamma)\to L^2(\Gamma)} = \left\|P^+_{NtD}\right\|_{L^2(\Gamma)\to H^1(\Gamma)}, \tag{2.16}$$

$$\left\|P^+_{NtD}\right\|_{H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)} \leq \left\|P^+_{NtD}\right\|_{L^2(\Gamma)\to H^1(\Gamma)}. \tag{2.17}$$

Proof of Lemma 2.3. By interpolation, the bound (2.15) follows from the bound (2.14), and similarly (2.17) follows from (2.16); see, e.g., [32, Theorems B.2 and B.11]. We now prove (2.14); the proof of (2.16) is very similar. To prove (2.14), first note that, for $\phi \in L^2(\Gamma)$,

$$\left\|P^+_{DtN}\phi\right\|_{H^{-1}(\Gamma)} = \sup_{\psi\in H^1(\Gamma)\setminus\{0\}} \frac{\left|\langle P^+_{DtN}\phi, \psi\rangle_\Gamma\right|}{\|\psi\|_{H^1(\Gamma)}}, \tag{2.18}$$

where, in this proof, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the *real* duality pairing between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$ for $|s| \leq 1$ (i.e., $\langle \phi, \psi\rangle_\Gamma = \int_\Gamma \phi\,\psi\,\mathrm{d}s$ when $\phi, \psi \in L^2(\Gamma)$).

Using the radiation condition (1.4) and Green's second identity (which can be obtained from two copies of Green's first identity (2.6) with the roles of u and v interchanged in the second one), one can show that

$$\langle P^+_{DtN}\psi, \phi\rangle_\Gamma = \langle P^+_{DtN}\phi, \psi\rangle_\Gamma \tag{2.19}$$

for $\phi \in H^{1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$. (Note that the fact that $\langle\cdot,\cdot\rangle_\Gamma$ is the *real* duality pairing is crucial; see [54, Lemma 4.10].) By the density of $H^{1/2}(\Gamma)$ in $L^2(\Gamma)$, (2.19) holds for $\phi \in L^2(\Gamma)$ and $\psi \in H^1(\Gamma)$. Therefore, (2.18) and (2.19) imply that, for $\phi \in L^2(\Gamma)$,

$$\left\|P^+_{DtN}\phi\right\|_{H^{-1}(\Gamma)} = \sup_{\psi\in H^1(\Gamma)\setminus\{0\}} \frac{\left|\langle P^+_{DtN}\psi, \phi\rangle_\Gamma\right|}{\|\psi\|_{H^1(\Gamma)}} \leq \left\|P^+_{DtN}\right\|_{H^1(\Gamma)\to L^2(\Gamma)} \|\phi\|_{L^2(\Gamma)},$$

and thus $\left\|P^+_{DtN}\right\|_{L^2(\Gamma)\to H^{-1}(\Gamma)} \leq \left\|P^+_{DtN}\right\|_{H^1(\Gamma)\to L^2(\Gamma)}$. A similar argument shows the reverse inequality, and thus we have proved (2.14). ∎

3. Exterior DtN and NtD bounds for Helmholtz (Theorems 1.4 and 1.5).

Overview of the proofs of Theorems 1.4 *and* 1.5. Following [28], we reduce the problem of bounding the DtN and NtD maps for solutions of $\Delta u + k^2 u = 0$ in Ω_+ to

1. bounding the solution of $\Delta u + k^2 u = -f$ in Ω_+ with zero Dirichlet or Neumann boundary conditions, and
2. bounding solutions of $\Delta v - \lambda^2 v = 0$ in Ω_+ in terms of their Dirichlet and Neumann traces.

The bounds for the first task are given by the resolvent estimates summarized in section 3.1. The bounds for the second task are given in section 3.2.

For the bounds on the DtN and NtD maps when $\gamma_+ u \in H^1(\Gamma)$ and $\partial_n^+ u \in L^2(\Gamma)$, we need to use bounds originally proven by Nečas on the solutions of second order strongly elliptic systems. Proofs of these bounds in the general case can be found in [47, sections 5.1.2 and 5.2.1] and [32, Theorem 4.24]. We prove them for the Helmholtz equation in section 3.3, however, since we need to keep track of how the constants depend on k (and this is not done in [47, section 5] and [32, Theorem 4.24]).

3.1. Summary of resolvent estimates.

The following resolvent estimates are key components in the proofs of Theorems 1.4 and 1.5.

THEOREM 3.1 (resolvent estimates). *Let $f \in L^2(\Omega_+)$ have compact support, and let $u \in H^1_{\rm loc}(\Omega_+)$ be a solution to the Helmholtz equation $\Delta u + k^2 u = -f$ in Ω_+ that satisfies the Sommerfeld radiation condition* (1.4). *If* either

(a) *Ω_+ is a two- or three-dimensional nontrapping domain (in the sense of Definition* 1.1*) and one of $\gamma_+ u$ and $\partial_n^+ u$ equals zero,* or

(b) *Ω_- is a nontrapping polygon (in the sense of Definition* 1.2*) and one of $\gamma_+ u$ and $\partial_n^+ u$ equals zero,* or

(c) *Ω_- is a two- or three-dimensional Lipschitz domain that is star-shaped (in the sense of Definition* 1.3(i)*) and $\gamma_+ u = 0$,*

then, given $k_0 > 0$ and $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$,

$$\|\nabla u\|_{L^2(\Omega_R)} + k\,\|u\|_{L^2(\Omega_R)} \lesssim \|f\|_{L^2(\Omega_+)} \tag{3.1}$$

for all $k \geq k_0$ (where the omitted constant depends on k_0 and R).

References for the proof of Theorem 3.1.

(a) The bound (3.1) was proved in [58, Theorem 7] under a condition [58, Condition D′] about the propagation of singularities that was later proved to hold when Ω_+ is nontrapping in [36], [37]. (Note that for these geometries, we also have that $u \in H^2(\Omega_R)$, and then $\|u\|_{H^2(\Omega_R)} \lesssim k\,\|f\|_{L^2(\Omega_+)}$ by, e.g., combining the bound (3.1) with [22, Theorem 2.3.3.2].) The bound (3.1) in the case of zero Dirichlet boundary conditions was also proved in [44, Theorem I.2D] (using the vector field constructed in [46, section 4]) when Ω_+ is a two-dimensional nontrapping domain and the curvature of Γ does not change sign infinitely often.

(b) The bound (3.1) was proved when Ω_- is a nontrapping polygon in [8, Corollary 3] using Vainberg's argument and the propagation of singularities results in [40]. (Note that [8, Corollary 3] proves that $k\|u\|_{L^2(\Omega_R)} \lesssim \|f\|_{L^2(\Omega_+)}$, but then the bound (3.1) follows by using part (a) of Lemma 2.2.)

(c) The bound (3.1) was proved when Ω_- is a star-shaped Lipschitz domain in 2- or 3-d in [12, Lemma 3.5]. (Actually [12, Lemma 3.5] only proved the result for C^∞ star-shaped domains, but the density result in [42, Appendix A] means that the proof works for Lipschitz star-shaped domains; see Remark 3.8). To obtain this result, Chandler-Wilde and Monk used the identity that arises from the multiplier (1.21) and then used a certain inequality [12, Lemma 2.1] (proved using the asymptotics of Bessel and Hankel functions) to deal with the contribution from infinity; in [55, Lemma 2.4] it is shown that this inequality can also be proved using the identity arising from Morawetz–Ludwig multiplier (1.18). We note that, using a limiting argument, the bound (3.1) was then established for star-shaped domains with no assumption on the smoothness of Γ, only the assumption that if $\mathbf{x} \in \Omega_+$, then $s\mathbf{x} \in \Omega_+$ for every $s > 1$ [12, Lemma 3.8] (and thus this second result contains the result for Lipschitz star-shaped domains as a special case). For our DtN and NtD bounds, we need Γ to be Lipschitz (so that $\partial_n^+ u$ is well-defined), and thus we cannot use this more general result. □

Remark 3.2 (bounds on the inf-sup constant). It is a standard result that, given a variational problem, a bound on the solution in terms of the data is equivalent to a lower bound on the inf-sup constant; see, e.g., [51, Theorem 2.1.44] or [26, Theorem 2.15 and Remark 2.20]. Therefore, a corollary of Theorem 3.1 is that, under the geometric conditions in the theorem, the inf-sup constants of the standard variational formulations of the exterior Dirichlet and Neumann problems are $\gtrsim 1/k$. (The standard variational formulation of the Dirichlet problem is given by, e.g., [12, equations (3.3) and (3.4)] or [48, equation (2.6.146)], and the standard variational formulation of the Neumann problem is given by, e.g., [48, equation (2.6.147)] or [26, equation (3.1.5)]). The proof of this result for the Dirichlet problem is given in [12, Lemmas 3.3 and 3.4]; the proof for the Neumann problem is identical.

3.2. Bounds on the solutions of the modified Helmholtz equation.

LEMMA 3.3 (bounds on the solutions of modified Helmholtz in terms of their Dirichlet and Neumann traces). *Let $\Omega_\pm$ be as in section* 2. *If $v \in H^1_{\rm loc}(\Omega_+)$ satisfies $\Delta v - \lambda^2 v = 0$, then, given $\lambda_0 > 0$,*

$$\|\nabla v\|_{L^2(\Omega_+)} + \lambda \|v\|_{L^2(\Omega_+)} \lesssim \lambda^{1/2} \|\gamma_+ v\|_{H^{1/2}(\Gamma)} \tag{3.2}$$

for all $\lambda \geq \lambda_0$. Furthermore, if $\partial_n^+ v \in L^2(\Gamma)$, then, given $\lambda_0 > 0$,

$$\|\nabla v\|_{L^2(\Omega_+)} + \lambda \|v\|_{L^2(\Omega_+)} \lesssim \lambda^{-1/2} \left\|\partial_n^+ v\right\|_{L^2(\Gamma)} \tag{3.3}$$

for all $\lambda \geq \lambda_0$.

Proof of (3.3) *and references for the proof of* (3.2). The bound (3.2) is proved in [6, Proposition 1] (see also [52, Proposition 2.5.1]).

The inequality (3.3) can be proved using Green's first identity (2.6) as follows. Since v and ∇v are in $L^2(\Omega_+)$, we can apply Green's first identity (2.6) in Ω_+, and this yields

$$\int_{\Omega_+} \big(|\nabla v|^2 + \lambda^2 |v|^2\big) \mathrm{d}\mathbf{x} = -\langle \partial_n^+ v, \gamma_+ v\rangle_\Gamma.$$

If $\partial_n^+ v \in L^2(\Gamma)$, then

$$\|\nabla v\|^2_{L^2(\Omega_+)} + \lambda^2 \|v\|^2_{L^2(\Omega_+)} \leq \left\|\partial_n^+ v\right\|_{L^2(\Gamma)} \|\gamma_+ v\|_{L^2(\Gamma)}. \tag{3.4}$$

The multiplicative trace inequality (2.4) and Cauchy's inequality (2.7) imply that

$$\|\gamma_+ v\|^2_{L^2(\Gamma)} \lesssim \lambda^{-1} \Big(\|\nabla v\|^2_{L^2(\Omega_+)} + \lambda^2 \|v\|^2_{L^2(\Omega_+)}\Big), \tag{3.5}$$

and then using (3.5) in the right-hand side of (3.4), we obtain (3.3). □

Remark 3.4 (sharpness of the bounds in Lemma 3.3). The bounds (3.2) and (3.3) are sharp in their λ-dependence in both 2- and 3-d.

Indeed, when Ω_- is the unit ball there exists a $v^{(1)} \in H^1(\Omega_+)$ satisfying $\Delta v^{(1)} - \lambda^2 v^{(1)} = 0$ and

$$\left\|\partial_n^+ v^{(1)}\right\|_{H^s(\Gamma)} \sim \lambda \left\|\gamma_+ v^{(1)}\right\|_{H^t(\Gamma)} \tag{3.6}$$

for any s and t. (This can be shown using almost identical arguments to those in Lemma 3.12.) Using the definition of the normal derivative, one can show that the bound (3.2) implies the bound

$$\left\|\partial_n^+ v\right\|_{H^{-1/2}(\Gamma)} \lesssim \lambda \left\|\gamma_+ v\right\|_{H^{1/2}(\Gamma)}; \tag{3.7}$$

see [29, Lemma 15] and [52, Proposition 2.5.2]. The asymptotics (3.6) then show that (3.7) is sharp, and thus so is (3.2). Finally, using (3.5) we see that (3.3) implies that $\lambda\|\gamma_+ v\|_{L^2(\Gamma)} \lesssim \|\partial_n^+ v\|_{L^2(\Gamma)}$, and thus the asymptotics (3.6) show that (3.3) is sharp.

3.3. DtN and NtD bounds modulo terms in the domain. In this section we prove the k-explicit version of Nečas' result [47, sections 5.1.2 and 5.2.1], [32, Theorem 4.24] applied to solutions of $\Delta u + k^2 u = -f$ in Ω_+, i.e., bounds on the DtN and NtD maps in $H^1(\Gamma)$–$L^2(\Gamma)$ with the H^1-norm of u in Ω_+ and the L^2-norm of f in Ω_+ appearing on the right-hand sides.

LEMMA 3.5 (DtN and NtD bounds in $H^1(\Gamma)$–$L^2(\Gamma)$ modulo terms in the domain). *Let $\Omega_\pm$ be as in section 2. Given $f \in L^2(\Omega_+)$ with compact support, let $u \in H^1_{\rm loc}(\Omega_+, \Delta)$ be a solution to $\Delta u + k^2 u = -f$, and let $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$.*

(i) *If $\gamma_+ u \in H^1(\Gamma)$, then $\partial_n^+ u \in L^2(\Gamma)$ and*

$$\left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)} \lesssim \left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} + \left\|\nabla u\right\|^2_{L^2(\Omega_R)} + k^2 \left\|u\right\|^2_{L^2(\Omega_R)} + \left\|f\right\|^2_{L^2(\Omega_+)}. \tag{3.8}$$

(ii) *If $\partial_n^+ u \in L^2(\Gamma)$, then $\gamma_+ u \in H^1(\Gamma)$ and*

$$\begin{aligned}\left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} \lesssim{}& \left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)} + k^2 \left\|\gamma_+ u\right\|^2_{L^2(\Gamma)} \\ &+ \left\|\nabla u\right\|^2_{L^2(\Omega_R)} + k^2 \left\|u\right\|^2_{L^2(\Omega_R)} + \left\|f\right\|^2_{L^2(\Omega_+)}.\end{aligned} \tag{3.9}$$

To prove Lemma 3.5, we use a Rellich-type identity and its integrated form (Lemmas 3.6 and 3.7, respectively).

LEMMA 3.6 (a Rellich-type identity). *Let v be a complex-valued C^2 function on some set $D \subset \mathbb{R}^d$, let $\mathcal{L}v := \Delta v + k^2 v$, and let $\mathbf{Z} \in (C^1(D))^d$ be* real-*valued. Then, with the summation convention,*

$$\begin{aligned}2\Re\big(\overline{\mathbf{Z}\cdot\nabla v}\mathcal{L}v\big) ={}& \nabla\cdot\Big[2\Re\big(\overline{\mathbf{Z}\cdot\nabla v}\nabla v\big) + (k^2|v|^2 - |\nabla v|^2)\mathbf{Z}\Big] \\ &+ (\nabla\cdot\mathbf{Z})\big(|\nabla v|^2 - k^2|v|^2\big) - 2\Re\big(\partial_j \mathbf{Z}_i \partial_i v \overline{\partial_j v}\big).\end{aligned} \tag{3.10}$$

Proof. Expand the divergence on the right-hand side. (See, e.g., [55, Lemma 2.1] for more details.) ∎

Identities of the form $\overline{\mathcal{M}v}\mathcal{L}v$, where $\mathcal{M}v$ is a derivative of v, are associated with the name of Rellich due to Rellich's introduction of the multiplier $\mathcal{M}v = \mathbf{x}\cdot\nabla v$ in [50]. Multipliers consisting of linear combinations of derivatives of v and v itself were introduced by Morawetz for the wave equation in [43] and the Helmholtz equation in [45], [44]. (See [55, Remark 2.7] for more bibliographic details.)

LEMMA 3.7 (integrated version of the Rellich-type identity). *For D a Lipschitz domain, define the space V by*

$$V := \Big\{ v : v \in H^1(D),\ \Delta v \in L^2(D),\ \gamma v \in H^1(\partial D),\ \partial_n v \in L^2(\partial D) \Big\}. \tag{3.11}$$

If $\mathbf{Z} \in (C^1(D))^d$ is real-*valued and $v \in V$, then*

$$\int_D \bigg(2\Re\big(\overline{\mathbf{Z}\cdot\nabla v}\,\mathcal{L}v\big) - (\nabla\cdot\mathbf{Z})\big(|\nabla v|^2 - k^2|v|^2\big) + 2\Re\big(\partial_i \mathbf{Z}_j \partial_i v \overline{\partial_j v}\big)\bigg)\mathrm{d}\mathbf{x}$$
$$= \int_{\partial D} \Big[2\Re\big(\overline{\mathbf{Z}\cdot\nabla v}\,\partial_n v\big) + \big(k^2|\gamma v|^2 - |\nabla v|^2\big)(\mathbf{Z}\cdot\mathbf{n})\Big]\mathrm{d}s, \tag{3.12}$$

where the expression ∇v in the integral on ∂D is understood as $\nabla_\Gamma(\gamma v) + \mathbf{n}\partial_n v$, and $\mathbf{n}$ is the outward-pointing, unit, normal vector to D.

Proof. This is a consequence of the divergence theorem applied to the identity (3.10). The divergence theorem $\int_D \nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial D}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s$ is valid when Ω is Lipschitz and $\mathbf{F} \in (C^1(\overline{D}))^d$ [32, Theorem 3.34]. In [42, Appendix A] it is proved that $\mathcal{D}(\overline{D}) := \{U|_D : U \in C^\infty(\mathbb{R}^d)\}$ is dense in V, and thus (3.12) holds for any $v \in V$. ∎

Proof of Lemma 3.5. The fact that $\gamma_+ u \in H^1(\Gamma)$ implies that $\partial_n^+ u \in L^2(\Gamma)$, and vice versa, was proved by Nečas in [47, sections 5.1.2 and 5.2.1] (see also [32, Theorem 4.24]) using the identity (3.10). Instead of repeating Nečas' proof keeping track of the dependence on k, we use his regularity result to justify applying the integrated identity (3.12) in Ω_+.

If $R > \sup_{\mathbf{x}\in\Omega_-}|\mathbf{x}|$, then $u \in V_R$, where the space V_R is defined by (3.11) with D replaced by Ω_R. Indeed, (i) $u \in H^1_{\mathrm{loc}}(\Omega_+,\Delta)$ implies that $u \in H^1(V_R,\Delta)$, (ii) if $\gamma_+ u \in H^1(\Gamma)$, then $\partial_n^+ u \in L^2(\Gamma)$ by Nečas' regularity result and vice versa, and (iii) interior H^2-regularity of the Laplacian (see, e.g., [19, section 6.3.1, Theorem 1] or [32, Theorem 4.16]) implies that $\gamma u \in H^1(\Gamma_R)$ and $\partial u/\partial n \in L^2(\Gamma_R)$.

Since $u \in V_R$, the identity (3.12) holds with D replaced by Ω_R, v replaced by u, and $\mathbf{Z}$ any real-valued, C^1 vector field, i.e.,

$$\int_{\Gamma\cup\Gamma_R} (\mathbf{Z}\cdot\mathbf{n})\left(\left|\partial_n^+ u\right|^2 + k^2|\gamma_+ u|^2 - |\nabla_\Gamma(\gamma_+ u)|^2\right) + 2\Re\left(\mathbf{Z}\cdot\overline{\nabla_\Gamma(\gamma_+ u)}\partial_n^+ u\right)\mathrm{d}s$$
$$+ \int_{\Omega_R} 2\Re\big(\mathbf{Z}\cdot\overline{\nabla u}\,f\big) + (\nabla\cdot\mathbf{Z})\left(|\nabla u|^2 - k^2|u|^2\right) - 2\Re(\partial_i\mathbf{Z}_j\partial_i u\overline{\partial_j u})\,\mathrm{d}\mathbf{x} = 0. \tag{3.13}$$

We now choose $\mathbf{Z}$ to be such that (a) there exists a $c > 0$ such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Gamma}\mathbf{Z}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) \geq c$, and (b) $\operatorname{supp}(\mathbf{Z}) \subset B_R$ (and thus $\mathbf{Z} = 0$ on Γ_R); such a $\mathbf{Z}$ exists by, e.g., [22, Lemma 1.5.1.9].

Rearranging the identity (3.13) and then using the facts (a) and (b) above along with the Cauchy–Schwarz inequality, we obtain that

$$\left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)} \lesssim \|\nabla_\Gamma(\gamma_+ u)\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma(\gamma_+ u)\|_{L^2(\Gamma)}\left\|\partial_n^+ u\right\|_{L^2(\Gamma)}$$
$$+ \|\nabla u\|_{L^2(\Omega_R)}\|f\|_{L^2(\Omega_R)} + \|\nabla u\|^2_{L^2(\Omega_R)} + k^2\|u\|^2_{L^2(\Omega_R)}.$$

Using the Cauchy inequality (2.7) on the second and third terms on the right-hand side, we obtain the DtN bound (3.8). The NtD bound (3.9) follows from the identity (3.13) in a similar way. ∎

Remark 3.8 (bounds in Lipschitz star-shaped domains). The density result in [42, Appendix A] that was used in the proof of Lemma 3.7 shows that the identities arising from the multipliers (1.18) and (1.21) hold when the domains are Lipschitz. Therefore, the DtN bound (1.17) obtained by Morawetz and Ludwig in [45] and the bound on the interior impedance problem (1.22) obtained by Melenk in [33, Proposition 8.1.4] and Cummings and Feng in [15, Theorem 1] hold when Ω_- is Lipschitz and star-shaped with respect to a ball (in the sense of Definition 1.3(ii)).

3.4. Proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Let a be such that $a > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$ and let $\zeta \in C^\infty(\Omega_+)$ be such that

$$\zeta(\mathbf{x}) = 0 \quad \text{for} \quad |\mathbf{x}| > a+1 \quad \text{and} \quad \zeta(\mathbf{x}) = 1 \quad \text{for} \quad |\mathbf{x}| < a.$$

We consider $\gamma_+ u$ as known and define $v \in H^1(\Omega_+)$ as the solution of

$$\Delta v - \lambda^2 v = 0 \quad \text{in } \Omega_+ \quad \text{and} \quad \gamma_+ v = \gamma_+ u \quad \text{on } \Gamma$$

with $v(\mathbf{x}) \to 0$ as $r \to \infty$. Given v, we define $h \in L^2(\Omega_+)$ by

$$h := -(k^2+\lambda^2)\zeta v - v\Delta\zeta - 2\nabla\zeta\cdot\nabla v \tag{3.14}$$

(note that since ζ has compact support, so does h), and we then define $w \in H^1_{\mathrm{loc}}(\Omega_+)$ as the solution of

$$\Delta w + k^2 w = h \quad \text{in } \Omega_+ \quad \text{and} \quad \gamma_+ w = 0 \quad \text{on } \Gamma,$$

satisfying the Sommerfeld radiation condition (1.4).

The whole point of these definitions is that $\widetilde{u} := \zeta v + w$ is then a solution of the homogeneous Helmholtz equation satisfying the Sommerfeld radiation condition, and, furthermore, $\gamma_+\widetilde{u} = \gamma_+ u$. By uniqueness, $\widetilde{u} = u$, and thus we have expressed u in terms of a solution of the inhomogeneous Helmholtz equation with zero Dirichlet trace, i.e., w, and a solution of the homogeneous modified Helmholtz equation with nonzero Dirichlet trace, i.e., v. (This result can therefore be understood as a kind of "gluing" theorem.)

Using the triangle inequality and the resolvent estimate (3.1) we have that, given $k_0 > 0$,

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim \|\nabla v\|_{L^2(\Omega_R)} + k\|v\|_{L^2(\Omega_R)} + \|h\|_{L^2(\Omega_R)}$$

for all $k \geq k_0$. The definition of h, (3.14), implies that

$$\|h\|_{L^2(\Omega_+)} \lesssim \|\nabla v\|_{L^2(\Omega_+)} + (k^2+\lambda^2)\|v\|_{L^2(\Omega_+)}, \tag{3.15}$$

and thus

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim \|\nabla v\|_{L^2(\Omega_R)} + (k^2+\lambda^2)\|v\|_{L^2(\Omega_R)}. \tag{3.16}$$

Using the bound on the modified Helmholtz equation (3.2) and the fact that $\gamma_+ v = \gamma_+ u$, we have

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim \lambda^{1/2}\left(1 + \frac{k^2+\lambda^2}{\lambda}\right)\|\gamma_+ u\|_{H^{1/2}(\Gamma)}. \tag{3.17}$$

Choosing $\lambda = k$ minimizes the power of k in the factor in front of $\|\gamma_+ u\|_{H^{1/2}(\Gamma)}$; thus we obtain

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim k^{3/2}\|\gamma_+ u\|_{H^{1/2}(\Gamma)}. \tag{3.18}$$

We now use Lemma 3.5 (the k-explicit version of Nečas' result) to obtain the bounds from $H^1(\Gamma)$ to $L^2(\Gamma)$ (1.6) and (1.7), and we then use the interpolation result (2.15) from Lemma 2.3 to obtain the bound (1.5) from (1.6). Indeed, the bounds (3.8) and (3.18) imply that

$$\left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)} \lesssim \left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} + k^2 \left\|\gamma_+ u\right\|^2_{L^2(\Gamma)} + k^3 \left\|\gamma_+ u\right\|^2_{H^{1/2}(\Gamma)}. \tag{3.19}$$

If we use in (3.19) the fact that $\left\|\gamma_+ u\right\|_{H^{1/2}(\Gamma)} \lesssim \left\|\gamma_+ u\right\|_{H^1(\Gamma)}$, we obtain the bound (1.6). Alternatively, the interpolation result

$$\left\|\gamma_+ u\right\|^2_{H^{1/2}(\Gamma)} \lesssim \left\|\gamma_+ u\right\|_{L^2(\Gamma)} \left\|\gamma_+ u\right\|_{H^1(\Gamma)},$$

[32, Lemma B.1 and Theorem B.11], the norm equivalence (2.5), and the Cauchy inequality (2.7) imply that

$$\left\|\gamma_+ u\right\|^2_{H^{1/2}(\Gamma)} \lesssim \frac{1}{k}\left(\frac{1}{\varepsilon}\left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} + \left(\varepsilon k^2 + \frac{1}{\varepsilon}\right)\left\|\gamma_+ u\right\|^2_{L^2(\Gamma)}\right) \tag{3.20}$$

for any $\varepsilon > 0$. Using (3.20) in (3.19) yields

$$\begin{aligned}\left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)} &\lesssim \left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} + k^2 \left\|\gamma_+ u\right\|^2_{L^2(\Gamma)} \\ &\quad + k^2\left(\frac{1}{\varepsilon}\left\|\nabla_\Gamma(\gamma_+ u)\right\|^2_{L^2(\Gamma)} + \left(\varepsilon k^2 + \frac{1}{\varepsilon}\right)\left\|\gamma_+ u\right\|^2_{L^2(\Gamma)}\right).\end{aligned} \tag{3.21}$$

We now aim to make the right-hand side of (3.21) a multiple of the weighted-$H^1(\Gamma)$ norm squared. The choice $\varepsilon = 1$ minimizes the power of k in front of the weighted norm, and thus (3.21) becomes the result (1.7). □

Proof of Theorem 1.5. Our goal is again to define v and w so that $u = \zeta v + w$, but this time $\partial_n^+ u$ is considered as known. We therefore define $v \in H^1(\Omega_+)$ as the solution of

$$\Delta v - \lambda^2 v = 0 \quad \text{in } \Omega_+ \quad \text{and} \quad \partial_n^+ v = \partial_n^+ u \quad \text{on } \Gamma$$

with $v(\mathbf{x}) \to 0$ as $r \to \infty$. Given v, we define h again by (3.14), and $w \in H^1_{\mathrm{loc}}(\Omega_+)$ as the solution of

$$\Delta w + k^2 w = h \quad \text{in } \Omega_+ \quad \text{and} \quad \partial_n^+ w = 0 \quad \text{on } \Gamma,$$

satisfying the Sommerfeld radiation condition.

By using the bound on h (3.15) and the resolvent estimate (3.1), we again have that (3.16) holds.

Using the bound on v (3.3) in (3.16), we obtain

$$\left\|\nabla u\right\|_{L^2(\Omega_R)} + k\left\|u\right\|_{L^2(\Omega_R)} \lesssim \frac{1}{\lambda^{1/2}}\left(1 + \frac{k^2+\lambda^2}{\lambda}\right)\left\|\partial_n^+ u\right\|_{L^2(\Gamma)}.$$

When $\lambda = k$ this bound becomes

$$\left\|\nabla u\right\|_{L^2(\Omega_R)} + k\left\|u\right\|_{L^2(\Omega_R)} \lesssim k^{1/2}\left\|\partial_n^+ u\right\|_{L^2(\Gamma)}, \tag{3.22}$$

and then the multiplicative trace inequality (2.4) and the Cauchy inequality (2.7) imply that

$$k^2\left\|\gamma_+ u\right\|^2_{L^2(\Gamma)} \lesssim k\Big(\left\|\nabla u\right\|^2_{L^2(\Omega_R)} + k^2\left\|u\right\|^2_{L^2(\Omega_R)}\Big) \lesssim k^2\left\|\partial_n^+ u\right\|^2_{L^2(\Gamma)}. \tag{3.23}$$

Using the bounds (3.22) and (3.23) in (3.9) (the k-explicit version of Nečas' result), we obtain the bound (1.9). This bound implies that $\|\gamma_+ u\|_{H^1(\Gamma)} \lesssim k\|\partial_n^+ u\|_{L^2(\Gamma)}$, and then the interpolation result (2.17) from Lemma 2.3 implies the bound (1.8). ∎

Remark 3.9 (the difference between the argument here and the argument in [28]). As discussed in section 1, the paper [28] proves the bounds (1.20) on the DtN and NtD maps in the trace spaces when Ω_+ is nontrapping (in the sense of Definition 1.1). The main differences between our argument for these spaces and theirs are the following: (i) We use sharper bounds on the solution of the modified Helmholtz equation with Dirichlet boundary conditions ((3.2) instead of the bound $\|\nabla v\|_{L^2(\Omega_+)} + \lambda\|v\|_{L^2(\Omega_+)} \lesssim \lambda\|\gamma_+ v\|_{H^{1/2}(\Gamma)}$). (ii) We use the Nečas result (3.8) to bound $\|\partial_n^+ u\|_{L^2(\Gamma)}$, and then use interpolation to bound $\|\partial_n^+ u\|_{H^{-1/2}(\Gamma)}$, whereas [28] effectively uses the fact that

$$\left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} \leq \left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \lesssim \|u\|_{H^2(\Omega_R)} \tag{3.24}$$

and then uses the resolvent estimate on the H^2-norm. (We say "effectively" because [28] bounds $\|\partial_n^+ u\|_{H^{-1/2}(\Gamma)}$ by bounding $\|\partial_n^+ v\|_{H^{-1/2}(\Gamma)}$ and $\|\partial_n^+ w\|_{H^{-1/2}(\Gamma)}$, and the inequalities (3.24) are used to obtain a bound on $\|\partial_n^+ w\|_{H^{-1/2}(\Gamma)}$.)

3.5. How sharp are the DtN and NtD bounds? This section is devoted to proving the following lemma.

LEMMA 3.10 (sharpness of the DtN and NtD maps). *Let $d = 2$ or 3. If the bounds on the DtN map*

$$\begin{aligned} \left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)} &\lesssim A \left\|\gamma_+ u\right\|_{H^{1/2}(\Gamma)}, \\ \left\|\partial_n^+ u\right\|_{L^2(\Gamma)} &\lesssim B \left\|\gamma_+ u\right\|_{H^1(\Gamma)}, \qquad \textit{and} \\ \left\|\partial_n^+ u\right\|_{L^2(\Gamma)} &\lesssim C \left\|\nabla_\Gamma(\gamma_+ u)\right\|_{L^2(\Gamma)} + D\, k \left\|\gamma_+ u\right\|_{L^2(\Gamma)} \end{aligned} \tag{3.25}$$

hold for all nontrapping domains (in the sense of Definition 1.1*) or for all Lipschitz domains that are star-shaped (in the sense of Definition* 1.3(i)*), then*

$$A \gtrsim k, \quad B \gtrsim k, \quad C \gtrsim 1, \quad D \gtrsim 1.$$

If the bounds on the NtD map

$$\left\|\gamma_+ u\right\|_{H^{1/2}(\Gamma)} \lesssim E \left\|\partial_n^+ u\right\|_{H^{-1/2}(\Gamma)}$$

and

$$\Big(\left\|\nabla_\Gamma(\gamma_+ u)\right\|_{L^2(\Gamma)} + k \left\|\gamma_+ u\right\|_{L^2(\Gamma)}\Big) \lesssim F \left\|\partial_n^+ u\right\|_{L^2(\Gamma)}$$

hold for all nontrapping domains (in the sense of Definition 1.1*), or for all Lipschitz domains that are star-shaped (in the sense of Definition* 1.3(i)*), then*

$$E \gtrsim k^{1/3} \quad \textit{and} \quad F \gtrsim k^{1/3}.$$

COROLLARY 3.11. *The bound on the DtN map from $H^1(\Gamma)$ to $L^2(\Gamma)$ for two- and three-dimensional Ω_- that are Lipschitz and star-shaped with respect to a ball given by Morawetz and Ludwig in* [45] *(i.e.,* (3.25) *with $C \sim 1, D \sim 1$) is sharp.*

Lemma 3.10 is proved by considering the specific case of Ω_- the unit ball. (Note that in this section we use the notation that $a \ll b$ if $a/b \to 0$ as $k \to \infty$, and $a \gg b$ if $b \ll a$.)

LEMMA 3.12. *If $\Omega_- = B_1$ (the unit ball) in 2- or 3-d, then there exists a $u^{(1)} \in H^1_{\mathrm{loc}}(\Omega_+)$ that has $\partial_n^+ u^{(1)} \in L^2(\Gamma)$ and satisfies $\Delta u^{(1)} + k^2 u^{(1)} = 0$, the Sommerfeld* *radiation condition* (1.4), *and the asymptotics*

$$\big\|\partial_n^+ u^{(1)}\big\|_{H^{-1/2}(\Gamma)} \sim k\big\|\gamma_+ u^{(1)}\big\|_{H^{1/2}(\Gamma)}, \tag{3.26}$$

$$\big\|\partial_n^+ u^{(1)}\big\|_{L^2(\Gamma)} \sim k\big\|\gamma_+ u^{(1)}\big\|_{H^1(\Gamma)}, \tag{3.27}$$

$$\big\|\partial_n^+ u^{(1)}\big\|_{L^2(\Gamma)} \sim k\big\|\gamma_+ u^{(1)}\big\|_{L^2(\Gamma)} \quad \textit{and } \big\|\nabla_\Gamma(\gamma_+ u^{(1)})\big\|_{L^2(\Gamma)} = 0, \tag{3.28}$$

as $k\to\infty$.

Furthermore, given any increasing function of k, $\widetilde{D}(k)$, *there exists a* $u^{(2)} \in H^1_{\mathrm{loc}}(\Omega_+)$ *that has* $\partial_n^+ u^{(2)} \in L^2(\Gamma)$ *and satisfies* $\Delta u^{(2)} + k^2 u^{(2)} = 0$, *the Sommerfeld radiation condition* (1.4), *and the asymptotics*

$$\big\|\partial_n^+ u^{(2)}\big\|_{L^2(\Gamma)} \sim \big\|\nabla_\Gamma(\gamma_+ u^{(2)})\big\|_{L^2(\Gamma)} \quad \textit{and} \quad \big\|\nabla_\Gamma(\gamma_+ u^{(2)})\big\|_{L^2(\Gamma)} \gg \widetilde{D}(k) k\big\|\gamma_+ u^{(2)}\big\|_{L^2(\Gamma)}, \tag{3.29}$$

as $k\to\infty$.

Finally there exists a $u^{(3)} \in H^1_{\mathrm{loc}}(\Omega_+)$ *that has* $\partial_n^+ u^{(3)} \in L^2(\Gamma)$ *and satisfies* $\Delta u^{(2)} + k^2 u^{(2)} = 0$, *the Sommerfeld radiation condition* (1.4), *and the asymptotics*

$$\big\|\gamma_+ u^{(3)}\big\|_{H^{1/2}(\Gamma)} \sim k^{1/3}\big\|\partial_n^+ u^{(3)}\big\|_{H^{-1/2}(\Gamma)}, \tag{3.30}$$

$$\big\|\nabla_\Gamma(\gamma_+ u^{(3)})\big\|_{L^2(\Gamma)} + k\big\|\gamma_+ u^{(3)}\big\|_{L^2(\Gamma)} \sim k^{1/3}\big\|\partial_n^+ u^{(3)}\big\|_{L^2(\Gamma)}, \tag{3.31}$$

as $k\to\infty$.

Proof of Lemma 3.10 *using Lemma* 3.12. The asymptotics (3.26) imply that $A \gtrsim k$, the asymptotics (3.27) imply that $B \gtrsim k$, and the asymptotics (3.28) imply that $D \gtrsim 1$. The asymptotics (3.29) then imply that $C \gtrsim 1$. Note that, for this last implication, the arbitrary increasing function $\widetilde{D}(k)$ is needed in (3.29) since, although (3.28) implies that D in (3.25) must be $\gtrsim 1$, we cannot rule out the possibility that D grows with k. The second bound in (3.29) ensures that $C\|\nabla_\Gamma(\gamma_+ u^{(2)})\|_{L^2(\Gamma)}$ is the dominant term on the right-hand side of (3.25), regardless of any potential growth in D. Finally, the asymptotics (3.30) imply that $E \gtrsim k^{1/3}$, and the asymptotics (3.31) imply that $F \gtrsim k^{1/3}$. ∎

Proof of Lemma 3.12. We first consider the two-dimensional case. The functions u_m defined by

$$u_m(r,\theta) := \frac{H_m^{(1)}(kr)}{H_m^{(1)}(k)} \mathrm{e}^{\mathrm{i}m\theta}, \quad m\in\mathbb{Z},$$

are in $H^1_{\mathrm{loc}}(\Omega_+)$ and satisfy $\Delta u_m + k^2 u_m = 0$, the Sommerfeld radiation condition, and $\partial_n^+ u_m \in L^2(\Gamma)$. Furthermore, $\gamma_+ u_m(\theta) = \exp(\mathrm{i}m\theta)$ and

$$\partial_n^+ u_m(\theta) = \frac{\partial u_m}{\partial r}(1,\theta) = k\frac{H_m^{(1)\prime}(k)}{H_m^{(1)}(k)} \mathrm{e}^{\mathrm{i}m\theta}.$$

Define the Fourier transform of a function $f:[0,2\pi]\to\mathbb{C}$ by $\widehat{f}(n) := \int_0^{2\pi} \exp(-\mathrm{i}n\theta)\, f(\theta)\,\mathrm{d}\theta$. We then have that $\widehat{u_m(1,\theta)}(n) = 2\pi\,\delta_{nm}$, and the definition of Sobolev spaces on Γ in terms of the Fourier transform implies that $\|\gamma_+ u_m\|^2_{H^s(\Gamma)} \sim (1+m^2)^s$, $\|\nabla_\Gamma(\gamma_+ u_m)\|^2_{L^2(\Gamma)} \sim m^2$, and

$$\big\|\partial_n^+ u_m\big\|^2_{H^s(\Gamma)} \sim k^2 \frac{N_m^2(k)}{M_m^2(k)}(1+m^2)^s,$$

where

$$N_m(k) := \left|H_m^{(1)'}(k)\right|, \qquad M_m(k) := \left|H_m^{(1)}(k)\right|,$$

and $\sim$ is meant as in section 2 but with the omitted constant independent of k and m. As $k \to \infty$ with m fixed, $N_m(k) \sim M_m(k)$ [1, equations (9.2.28) and (9.2.30)]. Therefore, the bounds (3.26)–(3.28) hold with $u^{(1)} := u_0$.

To prove that there exists a $u^{(2)}$ satisfying (3.29), first note that $\|\nabla_\Gamma(\gamma_+ u_m)\|_{L^2(\Gamma)} \sim m$ and $\|\gamma_+ u_m\|_{L^2(\Gamma)} \sim k$. Therefore, to prove that (3.29) holds, we need to show that, given any $\widetilde{D}(k)$, there exists an m (as a function of k) such that

$$m \gg \widetilde{D}(k)k^2 \quad \text{and} \quad k^2\frac{N_m^2(k)}{M_m^2(k)} \sim m^2 \quad \text{as } k \to \infty. \tag{3.32}$$

We now use the uniform asymptotic expansions of $H_\nu^{(1)}(\nu z)$ and $H_\nu^{(1)'}(\nu z)$ as $\nu \to \infty$ (uniform for all $z \in (0,\infty)$), aiming to ultimately let $\nu = m$ and $z = k/m$. The condition that $m \gg \widetilde{D}(k)k^2$ as $k \to \infty$ for some increasing function $\widetilde{D}(k)$ certainly implies that $m \gg k$ as $k \to \infty$. Therefore, when looking at $H_\nu^{(1)}(\nu z)$ and $H_\nu^{(1)'}(\nu z)$, we are interested in the case that $z \to 0$.

Using the uniform asymptotic expansions of $H_\nu^{(1)}(\nu z)$ and $H_\nu^{(1)'}(\nu z)$ given by, e.g., [1, equations (9.3.37) and (9.3.45)] or [49, equations (10.20.6) and (10.20.9)], we find that

$$\frac{N_\nu^2(\nu z)}{M_\nu^2(\nu z)} \sim \left(\frac{1-z^2}{\zeta z^2}\right)\frac{\left|\frac{\mathrm{Ai}(\alpha)}{\nu^{4/3}}C_0(\zeta) + \mathrm{e}^{2\pi\mathrm{i}/3}\frac{\mathrm{Ai}'(\alpha)}{\nu^{2/3}}\right|^2}{\left|\frac{\mathrm{Ai}(\alpha)}{\nu^{1/3}} + \mathrm{e}^{2\pi\mathrm{i}/3}\frac{\mathrm{Ai}'(\alpha)}{\nu^{5/3}}B_0(\zeta)\right|^2} \quad \text{as } \nu \to \infty,$$

uniformly for $z \in (0,\infty)$, where $\alpha := \exp(2\pi\mathrm{i}/3)\nu^{2/3}\zeta$,

$$\frac{2}{3}\zeta^{3/2} := \log\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2},$$

$C_0(\zeta)$ is a function that $\sim \zeta^{1/2}$ when $z \to 0$ and $\nu \to \infty$, and $B_0(\zeta)$ is a function that $\sim -\zeta^{-1/2}$ when $z \to 0$ and $\nu \to \infty$. Since ζ and ν are both real, $\alpha \in \exp(2\pi\mathrm{i}/3)\mathbb{R}$. If $z \to 0$, then $\zeta(z) \sim [\log(1/z)]^{2/3}$, and then $\alpha \to \infty$ as $\nu \to \infty$ and $z \to 0$. The asymptotics of $\mathrm{Ai}(\alpha)$ and $\mathrm{Ai}'(\alpha)$ are then given by

$$\mathrm{Ai}(\alpha) \sim \frac{\mathrm{e}^{-\beta}}{2\sqrt{\pi}\alpha^{1/4}} \quad \text{and} \quad \mathrm{Ai}'(\alpha) \sim -\alpha^{1/2}\,\mathrm{Ai}(\alpha) \quad \text{as } \alpha \to \infty,$$

where $\beta := 2\alpha^{3/2}/3$ [49, equations (9.7.5) and (9.7.6)]. Using these asymptotics, and the fact that $\exp(2\pi\mathrm{i}/3)\alpha^{1/2} = -\nu^{1/3}\zeta^{1/2}$, we find that

$$\frac{N_\nu^2(\nu z)}{M_\nu^2(\nu z)} \sim \left(\frac{1-z^2}{\zeta z^2}\right)\left|\frac{C_0(\zeta)+\nu\zeta^{1/2}}{\nu+\zeta^{1/2}B_0(\zeta)}\right|^2. \tag{3.33}$$

Using the facts that $\zeta \to \infty$, $C_0(\zeta) \sim \zeta^{1/2}$, and $B_0(\zeta) \sim -\zeta^{-1/2}$ as $\nu \to \infty$ and $z \to 0$, we have that

$$\frac{N_\nu^2(\nu z)}{M_\nu^2(\nu z)} \sim \frac{1}{z^2}$$

as $\nu \to \infty$ and $z \to 0$. If we let $\nu = m$ and $z = k/m$, then this implies that

$$k^2 \frac{N_m^2(k)}{M_m^2(k)} \sim m^2$$

as $k \to \infty$ and $m \to \infty$ with $m \gg k$. Therefore, given any increasing function of k, $\widetilde{D}(k)$, if we choose m to be a function of k such that $m \gg \widetilde{D}(k)k^2$ and let $u^{(2)} := u_m$, then the asymptotics (3.32) (and thus also the asymptotics (3.29)) hold.

Finally, we let $u^{(3)} := u_k$. The definition of u_m above implies that

$$\frac{\|\gamma_+ u_m\|_{H^{1/2}(\Gamma)}}{\left\|\partial_n^+ u_m\right\|_{H^{-1/2}(\Gamma)}} \sim \frac{M_m(k)}{N_m(k)} \frac{(1+m^2)^{1/2}}{k}. \tag{3.34}$$

The asymptotics

$$N_k(k) \sim k^{-2/3} \quad \text{and} \quad M_k(k) \sim k^{-1/3} \tag{3.35}$$

[1, equations (9.3.31)–(9.3.34)], [49, equations (10.19.9) and (10.19.13)] then imply (3.30). Similarly, the asymptotics (3.35) also imply (3.31).

In the three-dimensional case the argument proceeds almost exactly as before with

$$u_{l,m}(r,\theta,\phi) := \frac{h_l^{(1)}(kr)}{h_l^{(1)}(k)} Y_{l,m}(\theta,\phi), \quad l \in \mathbb{Z}^+,\ m = -l, \dots, l,$$

where $Y_{l,m}(\theta,\phi)$ are the spherical harmonics defined by [49, equation (14.30.1)]. (Note that l now plays the role that m played in the two-dimensional case.) The asymptotics (3.26)–(3.28) are satisfied if $u^{(1)} := u_{0,0}$, the asymptotics (3.29) are satisfied if $u^{(2)} := u_{l,0}$ and l is taken to be $\gg \widetilde{D}(k)k^2$, and the asymptotics (3.30) and (3.31) are satisfied with $u^{(3)} := u_{k,0}$. $\square$

Remark 3.13 (How sharp is Babich's bound on the NtD map?). We have $\|\gamma_+ u_m\|_{L^\infty(\Gamma)} = 1$ and $\|\partial_n^+ u_m\|_{L^\infty(\Gamma)} = kN_m(k)/M_m(k)$. If $m = k$, then this last quantity $\sim k^{2/3}$ as $k \to \infty$, and therefore the bound (1.19) on the NtD map is at most $k^{1/6}$ away from being sharp.

4. Bounds on the interior and exterior impedance problems. In this section we prove Theorems 1.6 and 1.8. We go through the argument for Theorem 1.6 (which concerns the interior problem) in sections 4.1–4.2 and then outline the necessary modifications to prove Theorem 1.8 (which concerns the exterior problem) in section 4.3.

We begin by defining precisely what we mean when we say that a function u satisfies the interior impedance problem.

DEFINITION 4.1 (interior impedance problem). *Given a bounded Lipschitz domain $\Omega_- \subset \mathbb{R}^d$, $d = 2,3$, with boundary Γ, functions $f \in (H^1(\Omega_-))'$ and $g \in H^{-1/2}(\Gamma)$, and $\eta \in \mathbb{R} \setminus \{0\}$, we say that $u \in H^1(\Omega_-)$ satisfies the* interior impedance problem *if*

$$a(u,v) = F(v) \quad \textit{for all } v \in H^1(\Omega_-), \tag{4.1}$$

where

$$a(u,v) := \int_{\Omega_-} (\nabla u \cdot \overline{\nabla v} - k^2 u \overline{v})\, d\mathbf{x} - i\eta \int_\Gamma \gamma_- u\, \overline{\gamma_- v}\, ds \quad \textit{and} \quad F(v) := \langle f, v\rangle_{\Omega_-} + \langle g, \gamma_- v\rangle_\Gamma, \tag{4.2}$$

where $\langle \cdot,\cdot\rangle_{\Omega_-}$ and $\langle \cdot,\cdot\rangle_\Gamma$ denote the duality pairings on Ω_- and Γ, respectively.

Therefore, the condition that u satisfies the PDE and boundary conditions (1.10) in Theorem 1.6 is to be understood as u satisfying the variational problem (4.1).

Green's first identity can be used to show that if $\eta \in \mathbb{R}$, then the solution to the interior impedance problem is unique; see, e.g., [18, Example 2.1]. The sesquilinear form $a(\cdot,\cdot)$ satisfies a Gårding inequality, and then Fredholm theory gives the existence of a solution to the variational problem (4.1); see, e.g., [51, Theorem 2.1.60], [32, Theorem 2.34].

To prove Theorem 1.6, we use the argument that Esterhazy and Melenk used to prove the bound (1.24) (which is closely related to the argument that Feng and Sheen used to prove the bound (1.23)—see Remark 4.9). This argument consists of the following two steps.

Step 1. Bound the solution of the interior impedance problem with $f = 0$ in terms of g. To do this, use Green's integral representation and bounds on the integral operators to bound u in terms of its Cauchy data ($\partial_n^- u$ and $\gamma_- u$), and then bound the Cauchy data by g using Green's first identity.

Step 2. Convert the inhomogeneous problem (i.e., with $f \neq 0$) into a homogenous one by using the Newtonian potential. Then use bounds on the Newtonian potential (also known as *free resolvent estimates*) along with the bounds obtained in Step 1 to obtain a bound on the solution of the interior impedance problem with $f \neq 0$.

Our improved bounds in Theorem 1.6 are the result of improved layer-potential bounds in Step 1. For completeness we also give the (short) argument in Step 2, although it is identical to that appearing in [18, section 2.1]. Before we present these arguments, we sketch a proof of Corollary 1.7.

References for the proof of Corollary 1.7. The argument that shows that the bound (1.11) can be used to prove the bound (1.14) can be found in, e.g., [18, Theorem 2.5] or [12, text between Lemmas 3.3 and 3.4]. The result (1.15) about the inf-sup constant then follows from, e.g., [51, Theorem 2.1.44]. □

4.1. Bounds on the problem with $f = 0$ (Step 1). We begin by recalling the fairly well-known result that the Cauchy data of the solution to the interior impedance problem with $f = 0$ can be bounded in terms of the impedance data g. (This is given in [12, equations (4.26) and (4.27)] and [18, Lemma 2.2].)

Lemma 4.2. *If $u \in H^1(\Omega_-)$ satisfies the interior impedance problem of Definition* 4.1 *with $f = 0$ and $g \in L^2(\Gamma)$, then*

$$\left\|\partial_n^- u\right\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)} \qquad \text{and} \qquad \|\gamma_- u\|_{L^2(\Gamma)} \leq \frac{1}{|\eta|}\|g\|_{L^2(\Gamma)}. \tag{4.3}$$

Proof. Since $u \in H^1(\Omega_-, \Delta)$ we can apply Green's first identity (2.6) in Ω_- with $v = u$ and take the imaginary part to obtain

$$\Im \int_\Gamma \overline{\gamma_- u}\, \partial_n^- u \,\mathrm{d}s = 0. \tag{4.4}$$

Using the impedance boundary condition (which holds as an equation in $L^2(\Gamma)$ as a consequence of the variational problem (4.1) and the definition of the normal derivative) to express $\partial_n^- u$ in (4.4) in terms of $\gamma_- u$ and g yields

$$\eta \|\gamma_- u\|^2_{L^2(\Gamma)} + \Im \int_\Gamma \overline{\gamma_- u}\, g \,\mathrm{d}s = 0. \tag{4.5}$$

Then, using the Cauchy–Schwarz inequality on the second term in (4.5), we obtain the second bound in (4.3). Similarly, using the impedance boundary condition to express

$\gamma_- u$ in (4.4) in terms of $\partial_n^- u$ and g and then using the Cauchy–Schwarz inequality yields the first bound in (4.3). $\square$

We now recall some facts about layer potentials. For $\phi \in L^2(\Gamma)$, the single- and double-layer potentials are defined by

$$\mathcal{S}_k\phi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\mathrm{d}s(\mathbf{y}), \quad \mathcal{D}_k\phi(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y})\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \Gamma, \tag{4.6}$$

where $\Phi_k(\mathbf{x},\mathbf{y})$ is defined by (1.27).

If $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, $|s| \le 1/2$, and $k \ge 0$, then

$$\chi\mathcal{S}_k : H^{s-1/2}(\Gamma) \to H^{s+1}(\mathbb{R}^d) \quad \text{and} \quad \chi\mathcal{D}_k : H^{s+1/2}(\Gamma) \to H^{s+1}(\Omega_\pm).$$

For $|s| < 1/2$ these mapping properties can be obtained from Green's integral representation and mapping properties of the Newtonian potential; see [14, Theorem 1], [51, Theorem 3.1.16], or [32, Theorems 6.11 and 6.12]. To establish the properties in the limit cases of $s = \pm 1/2$, one needs the harmonic analysis results summarized in, e.g., [11, Theorems 2.15 and 2.16]. (Note that the mapping properties for $|s| < 1/2$ can be obtained from those for $s = \pm 1/2$ by interpolation.)

Lemma 4.3 (bounds on the single- and double-layer potentials for Lipschitz Γ). *Let $d = 2$ or 3. With $\mathcal{S}_k$ and $\mathcal{D}_k$ defined by* (4.6)*, if $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, then, given $k_0 > 0$,*

$$\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k^{-1/2} \quad \text{and} \quad \|\chi\mathcal{D}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k^{1/2} \tag{4.7}$$

for all $k > k_0$.

While this paper was being written, Han and Tacy [23] also investigated the wavenumber-dependence of the norms of the single- and double-layer potentials. By using results about quasimodes and their restrictions to the boundary, Han and Tacy proved sharper bounds than those in Lemma 4.3 in the case that Γ is piecewise smooth.

Lemma 4.4 (bounds on the single- and double-layer potentials for piecewise smooth Γ [23, Theorems 1.1 and 1.4]). *Assume that Γ is piecewise smooth. With $\mathcal{S}_k$ and $\mathcal{D}_k$ defined by* (4.6)*, if $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, then, given $k_0 > 0$,*

$$\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k^{-3/4} \quad \text{and} \quad \|\chi\mathcal{D}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim 1 \tag{4.8}$$

for all $k > k_0$.

Note that the bound on $\mathcal{S}_k$ in (4.8) is sharp if Γ contains a flat piece (in either 2- or 3-d) [23, section 4.1], and the the bound on $\mathcal{D}_k$ is sharp if Ω_- is a two-dimensional ball [23, section 4.2].

Remark 4.5 (comparison of the bounds in Lemma 4.3 with previously obtained bounds). In [20, Theorems 3.4 and 4.5 and Lemma 3.5] Feng and Sheen prove that

$$\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim 1 \quad \text{and} \quad \|\chi\mathcal{D}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim 1 + k \tag{4.9}$$

for all $k > 0$. These bounds are then used to prove the bound on the interior impedance problem (1.23). A consequence of [34, Theorems 4.1 and 4.2] is that, given $k_0 > 0$,

$$\|\chi\mathcal{S}_k\|_{H^{-1}(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k \quad \text{and} \quad \|\chi\mathcal{D}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k \tag{4.10}$$

for all $k \geq k_0$, and these are the bounds that Esterhazy and Melenk used to obtain (1.24). We note that, first, this involves using the generous estimate that $\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \leq \|\chi\mathcal{S}_k\|_{H^{-1}(\Gamma)\to L^2(\mathbb{R}^d)}$ and, second, that the novel decompositions introduced in [34] that (4.10) are consequences of are not designed to produce sharp norm bounds. Indeed, the decompositions in [34] split these operators into parts with finite regularity but k-independent norm bounds and parts that are strongly smoothing with k-explicit bounds for their derivatives; these properties are then key in the analysis of the *hp* boundary element method in [31].

Proof of Lemma 4.3. The idea of the proof is to obtain the bounds on $\mathcal{S}_k$ and $\mathcal{D}_k$ in (4.7) by using, first, the definition of these operators in terms of the Newtonian potential and, second, bounds on the Newtonian potential (the so-called free resolvent estimates). We begin by recalling some facts about the Newtonian potential and these estimates.

Given $f \in L^2(\mathbb{R}^d)$ with compact support, let $\mathcal{N}_k f$ be the Newtonian potential of f defined by

$$\mathcal{N}_k f(\mathbf{x}) := \int_{\mathbb{R}^d} \Phi_k(\mathbf{x},\mathbf{y})\, f(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in\mathbb{R}^d. \tag{4.11}$$

If χ_1 and χ_2 are both in $C^\infty_{\text{comp}}(\mathbb{R}^d)$, then $\chi_1\mathcal{N}_k\chi_2 : H^s(\mathbb{R}^d) \to H^{s+2}(\mathbb{R}^d)$ for any $s\in\mathbb{R}$ [32, Theorem 6.1]. We have that $(\Delta + k^2)\mathcal{N}_k(f) = -f$ [32, Theorems 6.1 and 9.4] and $\mathcal{N}_k f$ satisfies the Sommerfeld radiation condition (1.4). Furthermore, for any $R>0$ and $k_0>0$,

$$k^{-1}\|\mathcal{N}_k f\|_{H^2(\Omega_R)} + \|\mathcal{N}_k f\|_{H^1(\Omega_R)} + k\|\mathcal{N}_k f\|_{L^2(\Omega_R)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \tag{4.12}$$

for all $k\geq k_0$, where the omitted constant depends only on R and k_0. This bound is known as the free resolvent estimate ("free" in the sense that, compared to the resolvent estimate in Theorem 3.1, there is no obstacle) and was proved by Vainberg in [58, Theorems 3 and 4]. (For some discussion on the appearances of this type of estimate in the literature, see [11, Remark 5.9].)

The adjoint of $\mathcal{N}_k$, $\mathcal{N}_k'$ is defined by

$$\mathcal{N}_k' f(\mathbf{x}) := \int_{\mathbb{R}^d} \overline{\Phi_k(\mathbf{y},\mathbf{x})}\, f(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \mathbf{x}\in\mathbb{R}^d.$$

We have that $\mathcal{N}_k' f = \overline{\mathcal{N}_k \overline{f}}$, and so the estimate (4.12) holds also for $\mathcal{N}_k'$.

The definitions of the single- and double-layer potentials (4.6) imply that, for $\psi\in L^2(\Gamma)$ and $f\in C^\infty_{\text{comp}}(\mathbb{R}^d)$,

$$\big(\mathcal{S}_k\psi, f\big)_{\mathbb{R}^d} = \big(\psi, \gamma\mathcal{N}_k' f\big)_\Gamma \quad\text{and}\quad \big(\mathcal{D}_k\psi, f\big)_{\mathbb{R}^d} = \big(\psi, \partial_n\mathcal{N}_k' f\big)_\Gamma, \tag{4.13}$$

where $(\cdot,\cdot)_{\mathbb{R}^d}$ denotes the L^2-inner product on $\mathbb{R}^d$, and $(\cdot,\cdot)_\Gamma$ denotes the L^2-inner product on Γ; see [32, p. 202], [51, Definition 3.1.5]. (Note that the Dirichlet and Neumann traces in (4.13) can be taken to be those from either the interior or the exterior. This is because $\mathcal{N}_k' f$ and its derivative are continuous across Γ due to the mapping properties of $\mathcal{N}_k'$ and the fact that $f\in C^\infty_{\text{comp}}(\mathbb{R}^d)$.)

Using the first equation in (4.13), the Cauchy–Schwarz inequality, and the multiplicative trace inequality (2.2), we obtain that, with $\psi\in L^2(\Gamma)$ and χ and $f\in C^\infty_{\text{comp}}(\mathbb{R}^d)$,

$$\left|\big(\chi\mathcal{S}_k\psi, f\big)_{\mathbb{R}^d}\right| \leq \|\psi\|_{L^2(\Gamma)}\|\gamma\mathcal{N}_k' f\|_{L^2(\Gamma)} \lesssim \|\psi\|_{L^2(\Gamma)}\Big(\|\mathcal{N}_k' f\|_{L^2(\Omega_R)}\|\mathcal{N}_k' f\|_{H^1(\Omega_R)}\Big)^{1/2}$$

for any $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$. Using the resolvent estimate (4.12), this last inequality becomes

$$\left|\left(\chi\mathcal{S}_k\psi, f\right)_{\mathbb{R}^d}\right| \lesssim k^{-1/2} \|\psi\|_{L^2(\Gamma)} \|f\|_{L^2(\mathbb{R}^d)} . \tag{4.14}$$

The inequality (4.14) holds for all $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, and thus for all $f \in L^2(\mathbb{R}^d)$ by the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. Therefore, we have that

$$\|\chi\mathcal{S}_k\psi\|_{L^2(\mathbb{R}^d)} = \sup_{f\in L^2(\mathbb{R}^d), f\neq 0} \frac{\left|\left(\chi\mathcal{S}_k\psi, f\right)_{\mathbb{R}^d}\right|}{\|f\|_{L^2(\mathbb{R}^d)}} \lesssim \frac{1}{k^{1/2}} \|\psi\|_{L^2(\Gamma)} ,$$

and the bound on $\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)}$ in (4.7) follows.

Similarly, with $\psi \in L^2(\Gamma)$ and χ and $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$,

$$\left|\left(\chi\mathcal{D}_k\psi, f\right)_{\mathbb{R}^d}\right| \le \|\psi\|_{L^2(\Gamma)} \|\partial_n\mathcal{N}'_k f\|_{L^2(\Gamma)} . \tag{4.15}$$

Since $\mathcal{N}'_k f \in H^2(\mathbb{R}^d)$, we have $\partial_n\mathcal{N}'_k f = \mathbf{n}\cdot\gamma\nabla(\mathcal{N}'_k f)$, and then the multiplicative trace inequality (2.2) implies that

$$\|\partial_n\mathcal{N}'_k f\|_{L^2(\Gamma)} \lesssim \left(\|\mathcal{N}'_k f\|_{H^1(\Omega_R)} \|\mathcal{N}'_k f\|_{H^2(\Omega_R)} \right)^{1/2} \tag{4.16}$$

for any $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$. Using (4.16) and the resolvent estimate (4.12) in (4.15) we obtain that

$$\left|\left(\chi\mathcal{D}_k\psi, f\right)_{\mathbb{R}^d}\right| \lesssim k^{1/2} \|\psi\|_{L^2(\Gamma)} \|f\|_{L^2(\mathbb{R}^d)} ,$$

and then the bound on $\|\chi\mathcal{D}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)}$ in (4.7) follows. ∎

To prove the following lemma, we first use Green's integral representation and the bounds on the layer potentials given by (4.7) and (4.8) to bound the solution of the homogeneous Helmholtz equation in terms of its Cauchy data. We then use Lemma 4.2 to bound the Cauchy data by $\|g\|_{L^2(\Gamma)}$.

LEMMA 4.6. *Let $u \in H^1(\Omega_-)$ be the solution of the interior impedance problem of Definition* 4.1 *with $f = 0$ and $g \in L^2(\Gamma)$. Then, given $k_0 > 0$,*

$$\|\nabla u\|_{L^2(\Omega_-)} + k\|u\|_{L^2(\Omega_-)} \lesssim k^{1/2}\left(1 + \frac{k}{|\eta|}\right)\|g\|_{L^2(\Gamma)} \tag{4.17}$$

for all $k \ge k_0$. Furthermore, if Γ is piecewise smooth, then, given $k_0 > 0$,

$$\|\nabla u\|_{L^2(\Omega_-)} + k\|u\|_{L^2(\Omega_-)} \lesssim \left(k^{1/4} + \frac{k}{|\eta|}\right)\|g\|_{L^2(\Gamma)} \tag{4.18}$$

for all $k \ge k_0$.

Proof. Green's integral representation implies that $u = \mathcal{S}_k\partial_n^- u - \mathcal{D}_k\gamma_- u$ [32, Theorem 7.5], and then the bounds (4.7) on $\mathcal{S}_k$ and $\mathcal{D}_k$ imply that, given $k_0 > 0$,

$$\|u\|_{L^2(\Omega_-)} \lesssim k^{-1/2}\left(\left\|\partial_n^- u\right\|_{L^2(\Gamma)} + k\|\gamma_- u\|_{L^2(\Gamma)}\right) \tag{4.19}$$

for all $k \ge k_0$. The bounds (4.3) on the Cauchy data then imply that

$$\|u\|_{L^2(\Omega_-)} \lesssim k^{-1/2}\left(1 + \frac{k}{|\eta|}\right)\|g\|_{L^2(\Gamma)} , \tag{4.20}$$

which is the bound on $\|u\|_{L^2(\Omega_-)}$ in (4.17).

To obtain the bound on $\|\nabla u\|_{L^2(\Omega_-)}$ in (4.17), we apply Green's first identity (2.6) in Ω_- with $v = u$ and use the impedance boundary condition to obtain

$$\|\nabla u\|^2_{L^2(\Omega_-)} - k^2 \|u\|^2_{L^2(\Omega_-)} = \mathrm{i}\eta \|\gamma_- u\|^2_{L^2(\Gamma)} + \int_\Gamma g\, \overline{\gamma_- u}\, \mathrm{d}s.$$

Taking the real part of this equation, using the Cauchy–Schwarz inequality, and then using the second bound in (4.3), we obtain that

$$\|\nabla u\|^2_{L^2(\Omega_-)} \lesssim k^2 \|u\|^2_{L^2(\Omega_-)} + \frac{1}{|\eta|} \|g\|^2_{L^2(\Gamma)}.$$

Using the bound (4.20) in the term involving $\|u\|_{L^2(\Omega_-)}$, we obtain the bound on $\|\nabla u\|_{L^2(\Omega_-)}$ in (4.17) and hence the result (4.17) itself. The improved result (4.18) when Γ is piecewise smooth follows in a similar way by using the bounds (4.8) instead of (4.7). ∎

COROLLARY 4.7. *If u satisfies the interior impedance problem of Definition* 4.1 *with $f = 0$ and $g \in L^2(\Gamma)$, then, given $k_0 > 0$,*

$$\|\nabla_\Gamma(\gamma_- u)\|_{L^2(\Gamma)} \lesssim k^{1/2} \left(1 + \frac{k}{|\eta|}\right) \|g\|_{L^2(\Gamma)} \tag{4.21}$$

for all $k \geq k_0$. Furthermore, if Γ is piecewise smooth, then, given $k_0 > 0$,

$$\|\nabla_\Gamma(\gamma_- u)\|_{L^2(\Gamma)} \lesssim \left(k^{1/4} + \frac{k}{|\eta|}\right) \|g\|_{L^2(\Gamma)} \tag{4.22}$$

for all $k \geq k_0$.

Proof. Repeating the argument in the proof of Lemma 3.5 for Ω_- instead of Ω_+, we obtain the bound

$$\|\nabla_\Gamma(\gamma_- u)\|^2_{L^2(\Gamma)} \lesssim \left\|\partial_n^- u\right\|^2_{L^2(\Gamma)} + k^2 \|\gamma_- u\|^2_{L^2(\Gamma)} + \|\nabla u\|^2_{L^2(\Omega_-)} + k^2 \|u\|^2_{L^2(\Omega_-)} \tag{4.23}$$

(recalling that $f = 0$). The result (4.21) follows from (4.23) using the bounds on $\|\partial_n^- u\|_{L^2(\Gamma)}$ and $\|\gamma_- u\|_{L^2(\Gamma)}$ in (4.3) and the bounds on $\|\nabla u\|_{L^2(\Omega_-)}$ and $\|u\|_{L^2(\Omega_-)}$ in (4.17). The result (4.22) follows in a similar way by using the bound (4.18) instead of (4.17). ∎

Remark 4.8. If u satisfies the interior impedance problem with $f = 0$ and $g \in L^2(\Gamma)$ and Ω_- is star-shaped with respect to a ball (in the sense of Definition 1.3(ii)), then, given $k_0 > 0$,

$$\|\nabla_\Gamma(\gamma_- u)\|_{L^2(\Gamma)} \lesssim \left(1 + \frac{k}{|\eta|}\right) \|g\|_{L^2(\Gamma)} \tag{4.24}$$

for all $k \geq k_0$. (Note that our bound for general Lipschitz Ω_-, (4.21), is a factor of $k^{1/2}$ worse.)

The bound (4.24) can be proved in one of two ways. The first consists of using the bound on the solution in the domain (1.22) and the bounds on the Cauchy data (4.3) in the bound (4.23). The second consists of using the fact that, under the star-shapedness assumption, integrating the identity arising from the multiplier (1.21) over Ω_- shows that, given $k_0 > 0$,

$$\|\nabla_\Gamma(\gamma_- u)\|_{L^2(\Gamma)} \lesssim \left\|\partial_n^- u\right\|_{L^2(\Gamma)} + k \|\gamma_- u\|_{L^2(\Gamma)}$$

for all $k \geq k_0$, and then (4.24) follows by using the bounds (4.3). The bound (4.24) was proved in [12, proof of Lemma 4.5, equation (4.28)] via the second method.

4.2. Bounds on the problem with nonzero f (Step 2).

Proof of Theorem 1.6 *using Lemma* 4.6. The strategy is to reduce the problem with $f \neq 0$ into a problem with $f = 0$ and then use Lemma 4.6.

Given $f \in L^2(\Omega_-)$, let $u_0 := \mathcal{N}_k f$. The bound (4.12) then holds for u_0 with the norms on the left-hand side all on Ω_-. If $\widetilde{g} := g - (\partial_n^- u_0 - \mathrm{i}\eta\gamma_- u_0)$, then we have that $\widetilde{u} := u - u_0$ satisfies

$$\Delta\widetilde{u} + k^2\widetilde{u} = 0 \quad \text{in } \Omega_- \quad \text{and} \quad \partial_n^-\widetilde{u} - \mathrm{i}\eta\gamma_-\widetilde{u} = \widetilde{g} \quad \text{on } \Gamma.$$

Using the triangle inequality, the bound (4.17) for $\widetilde{u}$, and the resolvent estimate (4.12) for u_0, we obtain that

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim k^{1/2}\left(1 + \frac{k}{|\eta|}\right)\|\widetilde{g}\|_{L^2(\Gamma)} + \|f\|_{L^2(\Omega_-)}\,. \tag{4.25}$$

Therefore, we only need to bound $\|\widetilde{g}\|_{L^2(\Gamma)}$ in terms of $\|g\|_{L^2(\Gamma)}$ and $\|f\|_{L^2(\Omega_-)}$. The definition of $\widetilde{g}$ implies that

$$\|\widetilde{g}\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)} + \left\|\partial_n^- u_0\right\|_{L^2(\Gamma)} + |\eta|\,\|\gamma_- u_0\|_{L^2(\Gamma)}\,.$$

Since $u_0 \in H^2(\Omega_-)$, we have that $\partial_n^- u = \mathbf{n}\cdot\gamma(\nabla u_0)$, and then using the multiplicative trace inequality (2.2) and the resolvent estimate (4.12), we have

$$\begin{aligned}\|\widetilde{g}\|_{L^2(\Gamma)} &\lesssim \|g\|_{L^2(\Gamma)} + \Big(\|u_0\|_{H^1(\Omega_-)}\,\|u_0\|_{H^2(\Omega_-)}\Big)^{1/2} + |\eta|\,\Big(\|u_0\|_{L^2(\Omega_-)}\,\|u_0\|_{H^1(\Omega_-)}\Big)^{1/2} \\ &\lesssim \|g\|_{L^2(\Gamma)} + k^{1/2}\left(1 + \frac{|\eta|}{k}\right)\|f\|_{L^2(\Omega_-)}\,.\end{aligned}$$

Using this last bound in (4.25) yields the result (1.11). The improved result for piecewise smooth Γ comes from using the bound (4.18) instead of (4.17) at the beginning of the proof (to obtain an improved factor in front of $\|\widetilde{g}\|_{L^2(\Gamma)}$ in (4.25)). $\square$

Remark 4.9 (Why not just do everything from Green's integral representation with $f \neq 0$?). To prove Theorem 1.6, we first proved bounds on the interior impedance problem with $f = 0$ using Green's integral representation (resulting in the bound (4.17)) and then used bounds on the Newtonian potential, $\mathcal{N}_k$, to prove bounds on the interior impedance problem with $f \neq 0$.

Alternatively, we could start from Green's integral representation with $f \neq 0$,

$$u = \mathcal{S}_k\partial_n^- u - \mathcal{D}_k\gamma_- u + \mathcal{N}_k f,$$

and then use the bounds on $\mathcal{S}_k$, $\mathcal{D}_k$, and $\mathcal{N}_k$ given by (4.7) and (4.12), along with the impedance boundary condition, to obtain

$$\|u\|_{L^2(\Omega_-)} \lesssim k^{1/2}\left(1 + \frac{|\eta|}{k}\right)\|\gamma_- u\|_{L^2(\Gamma)} + \frac{1}{k^{1/2}}\|g\|_{L^2(\Gamma)} + \frac{1}{k}\|f\|_{L^2(\Omega_-)}\,. \tag{4.26}$$

The argument involving Green's first identity that led to the bounds (4.33) for the problem with $f = 0$ can be used to prove that

$$\frac{|\eta|}{2}\,\|\gamma_- u\|^2_{L^2(\Gamma)} \leq \frac{1}{2\varepsilon}\,\|f\|^2_{L^2(\Omega_-)} + \frac{\varepsilon}{2}\,\|u\|^2_{L^2(\Omega_-)} + \frac{1}{2|\eta|}\,\|g\|^2_{L^2(\Gamma)} \tag{4.27}$$

for any $\varepsilon > 0$, and then this bound can be used in (4.26) to prove a bound on u in terms of g and f. (This is exactly the method used in [20] with the bounds on $\mathcal{S}_k$ and

$\mathcal{D}_k$ (4.9) used instead of (4.7) and the weaker bound $\|\mathcal{N}_k f\|_{L^2(\Omega_-)} \lesssim \|f\|_{L^2(\Omega_-)}$ used instead of (4.12).)

When $|\eta| \gtrsim k$, this method results in a bound identical in its k- and η-dependence to (1.11). When $|\eta| \ll k$, this method yields a bound that is weaker than (1.11) in its k- and η-dependence.

LEMMA 4.10 (sharpness of the interior impedance bounds). *If the bound on the solution of the interior impedance problem with $f \in L^2(\Omega_-)$, $g \in L^2(\Gamma)$, and $\eta = \pm k$*

$$\|\nabla u\|_{L^2(\Omega_-)} + k\,\|u\|_{L^2(\Omega_-)} \lesssim A\,\|g\|_{L^2(\Gamma)} + B\,\|f\|_{L^2(\Omega_-)} \tag{4.28}$$

holds whenever Ω_- is a bounded Lipschitz domain in 2- or 3-d, then

$$A \gtrsim k^{-1/2} \quad \text{and} \quad B \gtrsim 1.$$

Lemma 4.10 is proved by combining the following two lemmas.

LEMMA 4.11. *If $\Omega_- = B_1$ (the unit ball) in 2- or 3-d, then there exists a $u^{(1)} \in H^1(\Omega_-)$ that has $\partial_n^- u^{(1)} \in L^2(\Gamma)$ and satisfies $\Delta u^{(1)} + k^2 u^{(1)} = 0$ and the asymptotics*

$$\left\|\partial_n^- u^{(1)} \pm \mathrm{i}k\,\gamma_- u^{(1)}\right\|_{L^2(\Gamma)} \sim k\left\|\gamma_- u^{(1)}\right\|_{L^2(\Gamma)} \quad \text{as } k \to \infty. \tag{4.29}$$

LEMMA 4.12. *If Ω_- is any bounded, Lipschitz domain, then there exists a $\widetilde{f} \in L^2(\Omega_-)$ such that, if u is the solution of the interior impedance problem of Definition 4.1 with $g = 0$ and $f = \widetilde{f}$, there exists a $k_0 > 0$ such that*

$$k\,\|u\|_{L^2(\Omega_-)} \gtrsim \left\|\widetilde{f}\right\|_{L^2(\Omega_-)} \tag{4.30}$$

for all $k \geq k_0$.

Proof of Lemma 4.10 using Lemmas 4.11 and 4.12. The bound $B \gtrsim 1$ follows immediately from the bound (4.30) in Lemma 4.12. To prove the bound $A \gtrsim k^{-1/2}$, we consider the function $u^{(1)}$ of Lemma 4.11 and use the multiplicative trace inequality (2.2), the Cauchy inequality (2.7), and the bound (4.28) to obtain that

$$k^{1/2}\left\|\gamma_- u^{(1)}\right\|_{L^2(\Gamma)} \lesssim A\left\|\partial_n^- u^{(1)} \pm \mathrm{i}k\,\gamma_- u^{(1)}\right\|_{L^2(\Gamma)}. \tag{4.31}$$

Using the asymptotics (4.29) in (4.31), we obtain that $A \gtrsim k^{-1/2}$. ◻

Proof of Lemma 4.11. We first consider the two-dimensional case. The functions u_m defined by

$$u_m(r,\theta) := J_m(kr)\,\mathrm{e}^{\mathrm{i}m\theta}, \quad m \in \mathbb{Z},$$

are in $H^1(\Omega_-)$, satisfy $\Delta u_m + k^2 u_m = 0$, and have $\partial_n^+ u_m \in L^2(\Gamma)$. Furthermore, $\|\gamma_- u_m\|_{L^2(\Gamma)} \sim |J_m(k)|$ and

$$\left\|\partial_n^- u_m \pm \mathrm{i}k\,\gamma_- u_m\right\|_{L^2(\Gamma)} \sim k\sqrt{\left(J_m'(k)\right)^2 + \left(J_m(k)\right)^2}$$

as $k \to \infty$. When m is fixed, $|J_m'(k)| \sim |J_m(k)|$ [1, equations (9.2.5) and (9.2.11)], [49, equations (10.17.3) and (10.17.9)], and thus if $u^{(1)} := u_m$ for any (fixed) $m \in \mathbb{Z}$, then the asymptotics (4.29) hold.

In the three-dimensional case, the argument proceeds almost exactly as before with

$$u_{l,m}(r,\theta,\phi) := j_l(kr)\,Y_{l,m}(\theta,\phi), \quad l \in \mathbb{Z}^+,\ m = -l,\ldots,l.$$

We find that the asymptotics (4.29) are satisfied if $u^{(1)} := u_{l,0}$ for any (fixed) $l \in \mathbb{Z}^+$. □

Proof of Lemma 4.12. Given Ω_-, choose a $w \in C^\infty_{\mathrm{comp}}(\Omega_-)$ and define $\widetilde{f}$ by

$$\widetilde{f}(\mathbf{x}) := -\mathrm{e}^{\mathrm{i}kx_1}\left(\Delta w(\mathbf{x}) + 2\mathrm{i}k\frac{\partial w}{\partial x_1}(\mathbf{x})\right).$$

This definition implies that $\widetilde{u}(\mathbf{x}) := \mathrm{e}^{\mathrm{i}kx_1}w(\mathbf{x})$ satisfies $\Delta\widetilde{u}+k^2\widetilde{u} = -\widetilde{f}$ in Ω_-. Since $\widetilde{u}$ has compact support, $\partial_n^-\widetilde{u} - \mathrm{i}\eta\gamma_-\widetilde{u} = 0$ on Γ. Therefore, by uniqueness, the solution of the interior impedance problem, u, equals $\widetilde{u}$. The definition of $\widetilde{f}$ implies that

$$\|\widetilde{f}\|_{L^2(\Omega_-)} \lesssim \|\Delta w\|_{L^2(\Omega_-)} + k\left\|\frac{\partial w}{\partial x_1}\right\|_{L^2(\Omega_-)};$$

since both $\|\Delta w\|_{L^2(\Omega_-)}$ and $\|\partial w/\partial x_1\|_{L^2(\Omega_-)}$ are $\lesssim \|w\|_{L^2(\Omega_-)}$, and $\|u\|_{L^2(\Omega_-)} = \|w\|_{L^2(\Omega_-)}$, the bound (4.30) holds. □

The construction in Lemma 4.12 was used in [12, Lemma 3.10] to essentially prove that the resolvent estimate (3.1) under zero Dirichlet boundary conditions is sharp. We say "essentially" because actually [12, Lemma 3.10] proves that the bound $\alpha \gtrsim 1/k$ is sharp, where α is the inf-sup constant of the standard variational formulation of the exterior Dirichlet problem. However, since a lower bound on the inf-sup constant is equivalent to a resolvent estimate (see Remark 3.2) [12, Lemma 3.10] proves that the resolvent estimate for the exterior Dirichlet problem is sharp. Note that the argument as written in Lemma 4.12 can be easily modified to apply to the exterior Dirichlet, Neumann, or impedance problems (since any function in $C^\infty_{\mathrm{comp}}(\Omega_+)$ satisfies the radiation condition (1.4)).

4.3. Modifications needed to prove the bound on the exterior problem. As in the interior case, we begin by defining precisely what we mean when we say that u satisfies the exterior impedance problem.

Definition 4.13 (exterior impedance problem). *Given a bounded Lipschitz domain $\Omega_- \subset \mathbb{R}^d$, $d = 2, 3$, with boundary Γ, functions $f \in (H^1(\Omega_+))'$ and $g \in H^{-1/2}(\Gamma)$, and $\eta \in \mathbb{R}\setminus\{0\}$, fix $R > \sup_{\mathbf{x}\in\Omega_-}|\mathbf{x}|$ such that* $\operatorname{supp} f \subset B_R$. *Let $\Omega_R := \Omega_+ \cap B_R$ and let $\Gamma_R := \partial B_R$. We then say that $u \in H^1(\Omega_R)$ satisfies the* exterior impedance problem *if*

$$a(u, v) = F(v) \quad \textit{for all } v \in H^1(\Omega_R), \tag{4.32}$$

where

$$a(u, v) := \int_{\Omega_R}(\nabla u\cdot\overline{\nabla v} - k^2u\overline{v})\,\mathrm{d}\mathbf{x} - \mathrm{i}\eta\int_\Gamma \gamma_+u\,\overline{\gamma_+v}\,\mathrm{d}s - \big\langle T_R(\gamma u), \gamma v\big\rangle_{\Gamma_R}$$

and

$$F(v) := \langle f, v\rangle_{\Omega_-} - \langle g, \gamma_+v\rangle_\Gamma,$$

where $\langle\cdot,\cdot\rangle_{\Omega_-}$, $\langle\cdot,\cdot\rangle_\Gamma$, and $\langle\cdot,\cdot\rangle_{\Gamma_R}$ denote the duality pairings on Ω_-, Γ, and Γ_R, respectively, and T_R is the DtN operator on Γ_R. (See, e.g., [12, *equations* (3.5) *and* (3.6)], [48, *section* 2.6.3], *or* [35, *equations* (3.7) *and* (3.10)] *for the definition of T_R.)*

Given a $u \in H^1(\Omega_R)$ satisfying the exterior impedance problem of Definition 4.13, this u has a natural extension to a function in $H^1_{\mathrm{loc}}(\Omega_+)$. Indeed, with ${\Omega_R}^c :=$

$\mathbb{R}^d \setminus \Omega_R$ and $h_R := \gamma u$ on Γ_R, we extend u by setting $u|_{{\Omega_R}^c}$ to be the solution of the Dirichlet problem for the homogeneous Helmholtz equation in ${\Omega_R}^c$ satisfying the Sommerfeld radiation condition (1.4), with Dirichlet data on Γ_R equal to h_R. Using the variational problem (4.32), one can show that the Neumann traces on either side of Γ_R of the extended function are equal. The fact that both the Dirichlet and Neumann traces are continuous across Γ_R then implies that the extended function satisfies the homogeneous Helmholtz equation in a neighborhood of Γ_R (recalling that $\operatorname{supp} f \subset B_R$) and thus is C^∞ in this neighborhood.

Using this extension, one can prove that the solution to the variational problem (4.32) is unique (see, e.g., [13, Theorem 3.37], [11, Lemma 2.8]). The fact that $\Re(-\langle T_R\phi, \phi\rangle_{\Gamma_R}) \geq 0$ for all $\phi \in H^{1/2}(\Gamma_R)$ [48, Theorem 2.6.4] means that, just as in the interior case, $a(\cdot,\cdot)$ satisfies a Gårding inequality, and then Fredholm theory gives the existence of a solution to the variational problem (4.32).

To prove Theorem 1.8, we need the following lemma, which is the exterior analogue of Lemma 4.2 above. This result effectively appears in [27, Theorem 1].

LEMMA 4.14. *If $u \in H^1(\Omega_R)$ satisfies the exterior impedance problem of Definition 4.13 with $f = 0$ and $g \in L^2(\Gamma)$, then*

$$\left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \leq \|g\|_{L^2(\Gamma)} \quad \text{and} \quad \|\gamma_+ u\|_{L^2(\Gamma)} \leq \frac{1}{\eta}\|g\|_{L^2(\Gamma)}. \tag{4.33}$$

Proof. We extend u to ${\Omega_R}^c$ as described above. Since $u \in H^1_{\mathrm{loc}}(\Omega_+, \Delta)$, we can apply Green's first identity (2.6) with $v = u$ in $\Omega_{R'}$ for any $R' > \sup_{\mathbf{x}\in\Omega_-}|\mathbf{x}|$ and take the imaginary part to obtain

$$\Im \int_\Gamma \overline{\gamma_+ u}\, \partial_n^+ u \,\mathrm{d}s = \Im \int_{\Gamma_{R'}} \overline{\gamma u}\frac{\partial u}{\partial r}\,\mathrm{d}s \tag{4.34}$$

(remembering that $\mathbf{n}$ points into Ω_+). Using the fact that u satisfies the radiation condition, one can show that the right-hand side of (4.34) tends to $k\|F\|^2_{L^2(\mathbb{S}^{d-1})}$ as $R' \to \infty$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and F is the far-field pattern of u (see, e.g., [11, Lemmas 2.5 and 2.6]); therefore,

$$\Im \int_\Gamma \overline{\gamma_+ u}\, \partial_n^+ u \,\mathrm{d}s \geq 0. \tag{4.35}$$

Using the impedance boundary condition to express $\partial_n^+ u$ in (4.35) in terms of $\gamma_+ u$ and g yields

$$-\eta\|\gamma_+ u\|^2_{L^2(\Gamma)} + \Im \int_\Gamma \overline{\gamma_+ u}\, g\,\mathrm{d}s \geq 0,$$

and then using the Cauchy–Schwarz inequality on the second term gives us the second bound in (4.33). Similarly, using the impedance boundary condition to express $\gamma_+ u$ in (4.35) in terms of $\partial_n^+ u$ and g, and then using the Cauchy–Schwarz inequality, we obtain the first bound in (4.33). ∎

Proof of Theorem 1.8. As with the interior problem, we first consider the case $f = 0$. If u is the solution to the exterior impedance problem with $f = 0$, then Green's integral representation, $u = -\mathcal{S}_k\partial_n^+ u + \mathcal{D}_k\gamma_+ u$, holds; see, e.g., [32, Theorems 7.5 and 9.6]. Similar to the case of the interior problem, the bounds on the single- and double-layer potentials (4.7) then give

$$\|u\|_{L^2(\Omega_R)} \lesssim k^{-1/2}\Big(\left\|\partial_n^+ u\right\|_{L^2(\Gamma)} + k\|\gamma_+ u\|_{L^2(\Gamma)}\Big)$$

for any $R > \sup_{\mathbf{x}\in\Omega_-} |\mathbf{x}|$. (Note that, just as in the interior case, the bounds (4.8) can be used instead of (4.7) when Γ is piecewise smooth.) The bounds on the Cauchy data given by (4.33) then imply that

$$\|u\|_{L^2(\Omega_R)} \lesssim k^{-1/2}\left(1+\frac{k}{|\eta|}\right)\|g\|_{L^2(\Gamma)},$$

and using part (b) of Lemma 2.2 gives the bound on $\|\nabla u\|_{L^2(\Omega_R)}$. Therefore, the bound (4.17) holds with the norms on the left-hand side changed to be on $L^2(\Omega_R)$. The case when $f \neq 0$ follows in exactly the same way as for the interior problem, but now with every norm being in Ω_R. □

5. Proof of the bound on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$, Lemma 1.10. We now show how bounds on the exterior DtN and interior impedance-to-Dirichlet maps can be used to bound $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ (where $A'_{k,\eta}$ is the combined-field integral equation used to solve the exterior Dirichlet problem; see section 1.3).

Proof of Lemma 1.10. Since $A'_{k,\eta}$ is a bounded and invertible operator on $L^2(\Gamma)$ when $\eta \in \mathbb{R}\setminus\{0\}$ [11, Theorem 2.27], if we can show that $\|\phi\|_{L^2(\Gamma)} \le C\|A'_{k,\eta}\phi\|_{L^2(\Gamma)}$ for all $\phi \in L^2(\Gamma)$, then $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C$.

Given $\phi \in L^2(\Gamma)$, let $u := \mathcal{S}_k\phi$, where the *single-layer potential*, $\mathcal{S}_k$, is defined by

$$\mathcal{S}_k\phi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma.$$

The reason we do this is that the integral equation (1.25) arises from Green's integral representation (1.26), in which the solution of the BVP is expressed (modulo the known term u^I) as a single-layer potential with an unknown density. We also let $g := A'_{k,\eta}\phi$, so that (with this notation) we need to bound ϕ in terms of g.

Now, u is a solution of the Helmholtz equation in Ω_+ and Ω_- and satisfies the Sommerfeld radiation condition in Ω_+ [11, Theorem 2.14]. The jump relations for the single-layer potential are that

$$\gamma_\pm\mathcal{S}_k\phi = S_k\phi \quad \text{and} \quad \partial_n^\pm\mathcal{S}_k\phi = \left(\mp\frac{1}{2}I + D'_k\right)\phi \tag{5.1}$$

[32, Chapter 7], where the operators S_k and D'_k are defined by (1.29). The jump relations (5.1) and the definition of $A'_{k,\eta}$ (1.30) imply that

$$\partial_n^- u - \mathrm{i}\eta\gamma_- u = g, \tag{5.2}$$

$$\gamma_+ u = \gamma_- u \;\; \text{(and thus } \nabla_\Gamma(\gamma_+ u) = \nabla_\Gamma(\gamma_- u)), \quad \text{and} \tag{5.3}$$

$$\phi = \partial_n^- u - \partial_n^+ u. \tag{5.4}$$

By (5.2), u satisfies the interior impedance problem with data $g \in L^2(\Gamma)$. By (5.3), u satisfies the exterior Dirichlet problem with data given by the solution of the interior impedance problem. Given bounds on the solutions of the interior impedance and exterior Dirichlet problems, we can then use (5.4) to bound ϕ. Indeed, using (5.4), Lemma 4.2, and the DtN bound (1.32), we obtain that

$$\|\phi\|_{L^2(\Gamma)} \le \left\|\partial_n^- u\right\|_{L^2(\Gamma)} + \left\|\partial_n^+ u\right\|_{L^2(\Gamma)} \lesssim \left(1+\beta\frac{k}{|\eta|}\right)\|g\|_{L^2(\Gamma)} + \alpha\left\|\nabla_\Gamma(\gamma_+ u)\right\|_{L^2(\Gamma)}.$$

Then, using (5.3) and the impedance-to-Dirichlet bound (1.33), we find that

$$\|\phi\|_{L^2(\Gamma)} \lesssim \left(1 + \beta\frac{k}{|\eta|} + \alpha\delta\right) \|g\|_{L^2(\Gamma)} ,$$

which implies (1.34). $\square$

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