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Bounding acoustic layer potentials via oscillatory integral techniques

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Abstract We consider the Helmholtz single-layer operator (the trace of the single-layer potential) as an operator on $L^2(\Gamma)$ where Γ is the boundary of a 3–d obstacle. We prove that if Γ is C^2 and has strictly positive curvature then the norm of the single-layer operator tends to zero as the wavenumber k tends to infinity. This result is proved using a combination of (i) techniques for obtaining the asymptotics of oscillatory integrals, and (ii) techniques for obtaining the asymptotics of integrals that become singular in the appropriate parameter limit. This paper is the first time such techniques have been applied to bounding norms of layer potentials. The main motivation for proving this result is that it is a component of a proof that the combined-field integral operator for the Helmholtz exterior Dirichlet problem is coercive on such domains in the space $L^2(\Gamma)$.

Keywords Helmholtz equation · high frequency · boundary integral equation · layer potential · oscillatory integral operator

Mathematics Subject Classification (2000) 35J05 · 45M05 · 65N38

1 Introduction

Acoustic, electromagnetic, and elastic wave scattering problems with constant wave speed are often solved using integral equations. An important feature of the relevant integral operators in the frequency domain is that they are oscillatory, with the oscillation increasing as the wavenumber, k, increases. This paper considers a certain aspect of the k-explicit numerical analysis of Helmholtz boundary-integral equations; another paper in this special issue considering related aspects is [15].

This paper is concerned with how the norms of the integral operators associated with the Helmholtz equation behave as k increases. More precisely, we seek to prove k-explicit upper bounds on the norms of these operators that are valid when k is large. There already exist in the literature several k-explicit upper bounds on norms of Helmholtz integral operators, and we review these in detail in Section 1.2. We note at this stage, however, that all the currently-available k-explicit upper bounds on norms of Helmholtz integral operators fall into one of two categories:

1. Upper bounds when the obstacle is a ball (i.e. the boundary of the obstacle is the circle or sphere); these are obtained using the fact that the integral operators diagonalise in a basis of trigonometric polynomials (in 2–d) or spherical harmonics (in 3–d), with eigenvalues given explicitly in terms of Bessel and Hankel functions.
2. Upper bounds for general obstacles obtained using methods that *ignore* the oscillation in the integral operators.

In this paper, we consider the norm of the Helmholtz single-layer operator on 3–d, C^2 domains whose boundaries have strictly positive curvature, and we prove an upper bound that does not fit in either of these

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two categories. Indeed, the method we use to obtain this bound explicitly uses the fact that the relevant integral operator is highly oscillatory, and (as perhaps expected) the resulting bound is sharper than the corresponding one obtained using methods that ignore the oscillation. Note that, although we restrict attention to the single-layer operator on 3–d, C^2 domains with strictly positive curvature, the method we use is applicable to other operators and more general 2- and 3-d geometries.

1.1 Formulation of the problem

In this paper we only consider the sound-soft scattering problem for the Helmholtz equation (effectively the exterior Dirichlet problem), but the integral operators that arise in this problem also appear in formulations of other Helmholtz boundary value problems (BVPs); see, e.g., [7, §2.5–2.6].

Let $\Omega_- \subset \mathbb{R}^d$, with $d=2$ or 3, be a bounded Lipschitz open set with boundary $\Gamma := \partial\Omega_-$, such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $H^1_{\mathrm{loc}}(\Omega_+)$ denote the set of functions, v, such that v is locally integrable on Ω_+ and $\psi v \in H^1(\Omega_+)$ for every compactly supported $\psi \in C^\infty(\overline{\Omega_+}) := \{\psi|_{\Omega_+} : \psi \in C^\infty(\mathbb{R}^d)\}$. Let γ_+ denote the trace operator from Ω_+ to Γ. Let $\mathbf{n}$ be the outward-pointing unit normal vector to Ω_-, and let ∂_n^+ denote the normal derivative trace operator from Ω_+ to Γ that satisfies $\partial_n^+ u = \mathbf{n}\cdot\gamma_+(\nabla u)$ when $u \in H^2_{\mathrm{loc}}(\Omega_+)$. (We also call $\gamma_+ u$ the Dirichlet trace of u and $\partial_n^+ u$ the Neumann trace.)

Definition 1.1 (Sound-soft scattering problem) Given $k>0$ and an incident plane wave $u^I(\mathbf{x}) = \exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $|\widehat{\mathbf{a}}|=1$, find $u^S \in C^2(\Omega_+) \cap H^1_{loc}(\Omega_+)$ such that the total field $u := u^I + u^S$ satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_+, \qquad \gamma_+ u = 0 \quad \text{on } \Gamma,$$

and u^S satisfies the Sommerfeld radiation condition,

$$\frac{\partial u^S}{\partial r}(\mathbf{x}) - \mathrm{i}k u^S(\mathbf{x}) = o\left(\frac{1}{r^{(d-1)/2}}\right)$$

as $r := |\mathbf{x}| \to \infty$, uniformly in $\widehat{\mathbf{x}} := \mathbf{x}/r$.

It is well known that the solution to this problem exists and is unique; see, e.g., [7, Theorem 2.12].

The BVP in Definition 1.1 can be reformulated as an integral equation on Γ in two different ways. The first, the so-called *direct method*, uses Green's integral representation for the solution u, i.e.

$$u(\mathbf{x}) = u^I(\mathbf{x}) - \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\partial_n^+ u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Omega_+, \tag{1.1}$$

where $\Phi_k(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4}H_0^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big),\ d=2, \qquad \Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},\ d=3$$

(note that to obtain (1.1) from the usual form of Green's integral representation one must use the fact that u^I is a solution of the Helmholtz equation in Ω_-; see, e.g., [7, Theorem 2.43]).

Taking the Dirichlet and Neumann traces of (1.1) on Γ, one obtains two integral equations for the unknown Neumann boundary value $\partial_n^+ u$:

$$S_k \partial_n^+ u = \gamma_+ u^I, \qquad \left(\frac{1}{2}I + D_k'\right)\partial_n^+ u = \partial_n^+ u^I, \tag{1.2}$$

where the integral operators S_k and D_k', the single-layer operator and the adjoint-double-layer operator respectively, are defined for $\psi \in L^2(\Gamma)$ by

$$S_k\psi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad D_k'\psi(\mathbf{x}) := \int_\Gamma \frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma$$

(when Γ is Lipschitz, the integral defining D_k' is understood as a Cauchy principal value integral; see, e.g., [7, §2.3]).

Both integral equations in (1.2) fail to be uniquely solvable for certain values of k (for the first equation in (1.2) these are the k such that k^2 is a Dirichlet eigenvalue of the Laplacian in Ω_-, and for the second equation in (1.2) these are the k such that k^2 is a Neumann eigenvalue). The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation

$$A'_{k,\eta}\frac{\partial u}{\partial n}=f, \tag{1.3}$$

where

$$A'_{k,\eta}:=\frac{1}{2}I+D'_k-\mathrm{i}\eta S_k \tag{1.4}$$

is the *combined-potential* or *combined-field* operator, with $\eta\in\mathbb{R}\setminus\{0\}$ the so-called coupling parameter, and

$$f(\mathbf{x})=\partial_n^+u^I(\mathbf{x})-\mathrm{i}\eta\gamma_+u^I(\mathbf{x}),\quad \mathbf{x}\in\Gamma.$$

Since Ω_+ is Lipschitz, standard trace results imply that the unknown Neumann boundary value ∂_n^+u is in $H^{-1/2}(\Gamma)$. When Ω_+ is C^2, elliptic regularity implies that $\partial_n^+u\in L^2(\Gamma)$ (since $u\in H^2_{\mathrm{loc}}(\Omega_+)$), but $\partial_n^+u\in L^2(\Gamma)$ even when Ω_+ is Lipschitz via a regularity result of Nečas [22, §5.1.2], [19, Theorem 4.24 (ii)]. Therefore, even for Lipschitz Ω_+ we can consider the integral equation (1.3) as an operator equation in $L^2(\Gamma)$, which is a natural space for the practical solution of second-kind integral equations since it is self-dual. It is well known that, for $\eta\neq 0$, $A'_{k,\eta}$ is a bounded and invertible operator on $L^2(\Gamma)$ (see [7, Theorem 2.27]).

Instead of using Green's integral representation to formulate the BVP as an integral equation, one can pose the ansatz

$$u^S(\mathbf{x})=\int_\Gamma\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-\mathrm{i}\eta\int_\Gamma\Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y})$$

for $\phi\in L^2(\Gamma)$ and $\eta\in\mathbb{R}\setminus\{0\}$; this is the so-called *indirect method*. Imposing the boundary condition $\gamma_+u^S=-\gamma_+u^I$ on Γ leads to the integral equation

$$A_{k,\eta}\phi=-\gamma_+u^I, \tag{1.5}$$

where

$$A_{k,\eta}:=\frac{1}{2}I+D_k-\mathrm{i}\eta S_k, \tag{1.6}$$

and D_k is the double-layer operator, which is defined for $\psi\in L^2(\Gamma)$ by

$$D_k\psi(\mathbf{x})=\int_\Gamma\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),\quad \mathbf{x}\in\Gamma$$

(as with D'_k, the integral defining D_k is understood as a Cauchy principal value integral when Γ is Lipschitz).

Although the unknowns in the integral equations (1.3) and (1.5) are different, the identities

$$\int_\Gamma\phi\,S_k\psi\,\mathrm{d}s=\int_\Gamma\psi\,S_k\phi\,\mathrm{d}s\quad\text{and}\quad\int_\Gamma\phi\,D_k\psi\,\mathrm{d}s=\int_\Gamma\psi\,D'_k\phi\,\mathrm{d}s \tag{1.7}$$

for $\phi,\psi\in L^2(\Gamma)$ [7, Equation 2.37] mean that $A_{k,\eta}$ and $A'_{k,\eta}$ are adjoint with respect to the real-valued $L^2(\Gamma)$ inner product, and so in particular satisfy

$$\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}=\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

The second identity in (1.7) also implies that D_k and D'_k are adjoint with respect to the real-valued $L^2(\Gamma)$ inner product and satisfy

$$\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}=\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

The general question we consider in this paper is the following: how do $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ (and hence $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$) depend on k as k increases?

1.2 Summary of existing upper bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$

This paper is focused on obtaining k-explicit upper bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ that are valid when k is large. We note that several k-explicit lower bounds on these quantities were proved in [6] (see the review [7, §5.5.2] for an overview), and k-explicit upper bounds on these quantities that are sharp as $k\to 0$ were proved in [3, §2.6].

Here and in the rest of the paper, the notation $a\lesssim b$ means that $a\leq Cb$ for some constant $C>0$ that is independent of k (and any other parameters of interest).

1.2.1 Upper bounds when Γ is a circle or sphere

When Γ is a circle or sphere, both S_k and D_k diagonalise in a basis of trigonometric polynomials or spherical harmonics (see, e.g., [18, §3–4] or [11, Lemma 4.1] for the details). Bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ can be therefore be obtained by bounding the eigenvalues, which are given in terms of Bessel and Hankel functions. In [14], [11], and [2], the following upper bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ were obtained using this method.

Theorem 1.1 ([14], [11], [2]) *If Γ is a circle or sphere then, given $k_0 > 0$,*

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{-2/3} \quad \text{and} \quad \|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 \tag{1.8}$$

for all $k \geq k_0$. (Note that the omitted constants are independent of k but depend on k_0).

For more discussion of these results, see [7, Theorem 5.12].

1.2.2 Upper bounds for more general domains

The only currently-available upper bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ for domains other than the circle and sphere are the following.

Theorem 1.2 ([6, Theorems 3.3 and 3.5]) *If Γ is Lipschitz and $d = 2$ or 3, then*

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-3)/2} \quad \text{and} \quad \|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + k^{(d-1)/2} \tag{1.9}$$

for all $k > 0$.

These bounds can be proved using (at least) two different techniques:

1. the Riesz-Thorin interpolation theorem, and
2. Young's inequality for convolutions.

Young's inequality was used in [11, Lemma 4.14] to prove the bounds (1.9) for $d = 2$ when Γ is C^∞ (although, as we see below, the technique also works when Γ is Lipschitz and when $d = 3$). The Riesz-Thorin interpolation theorem was used in [6, Theorems 3.3 and 3.5] to prove the bounds (1.9) for $d = 2$ and $d = 3$ when Γ is Lipschitz.

In this paper, we also use the Riesz-Thorin method, and so we give an outline of this method below. We also give a brief outline of the method that uses Young's inequality, since this method is arguably the simplest way of obtaining the bounds (1.9), and this fact has perhaps not been fully appreciated before.

Overview of the Riesz-Thorin method. If T is an integral operator on Γ with kernel $t(\mathbf{x},\mathbf{y})$, i.e.,

$$T\phi(\mathbf{x}) = \int_\Gamma t(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

then, using the definitions of the L^1- and L^∞-operator norms, it is straightforward to show that

$$\|T\|_{L^1(\Gamma)\to L^1(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{y}\in\Gamma} \int_\Gamma \big|t(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{x}), \quad \text{and} \tag{1.10a}$$

$$\|T\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma} \int_\Gamma \big|t(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}) \tag{1.10b}$$

(provided these integrals exist). The Riesz-Thorin interpolation theorem implies that

$$\|T\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \Big(\|T\|_{L^1(\Gamma)\to L^1(\Gamma)}\Big)^{1/2}\Big(\|T\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)}\Big)^{1/2}$$

(see, e.g., [12, Theorem 6.27]), and thus a bound on $\|T\|_{L^2(\Gamma)\to L^2(\Gamma)}$ can be obtained by bounding the integrals on the right-hand sides of (1.10). In particular, if $|t(\mathbf{x},\mathbf{y})| \leq \widetilde{t}(\mathbf{x},\mathbf{y})$, where $\widetilde{t}$ is such that $\widetilde{t}(\mathbf{x},\mathbf{y}) = \widetilde{t}(\mathbf{y},\mathbf{x})$, then

$$\|T\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma} \int_\Gamma \widetilde{t}(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y}). \tag{1.11}$$

To obtain a bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, we can apply the bound (1.11) with $T = S_k$ and $\widetilde{t}(\mathbf{x},\mathbf{y})$ chosen as $|\Phi_k(\mathbf{x},\mathbf{y})|$. On the other hand, to obtain a bound on $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ we first write D_k as $D_0 + (D_k - D_0)$ and then apply (1.11) with $T = D_k - D_0$; we do this because the singularity of D_k is too strong for the operator itself to be bounded on $L^1(\Gamma)$ and $L^\infty(\Gamma)$ for general Lipschitz Γ (see, e.g., [6, Equation 3.8 onwards] for expressions for the kernels of D_k and $D_k - D_0$ in 2- and 3–d).

It is important to note that these bounds ignore the oscillation in k. For example, the method described above yields the bound

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma} \int_\Gamma \left|\Phi_k(\mathbf{x},\mathbf{y})\right| \mathrm{d}s(\mathbf{y}). \tag{1.12}$$

One can then use the bound

$$\left|\frac{\mathrm{i}}{4} H_0^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big)\right| \le \frac{1}{4}\sqrt{\frac{2}{\pi k|\mathbf{x}-\mathbf{y}|}} \tag{1.13}$$

(see, e.g., [6, Equation 1.22]) to obtain

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{-1/2} \quad \text{when } d=2. \tag{1.14}$$

Similarly, one can use the bound

$$\left|\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\right| \le \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{1.15}$$

to obtain

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 \quad \text{when } d=3. \tag{1.16}$$

Although the estimates (1.13) and (1.15) may appear crude, the resulting bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ for $d=2$, (1.14), is sharp when Γ contains a straight line segment (see [6, Theorem 4.2], [7, Lemma 5.18]). It is not yet known whether the bound for $d=3$, (1.16), is sharp for general Lipschitz Γ, and the main goal of this paper is to obtain sharper bounds than (1.16) for certain Γ (we discuss this more in §1.3).

Overview of the method using Young's inequality. Young's inequality for convolutions states that if $f \in L^p(\Gamma)$, $g \in L^q(\Gamma)$, with $1 \le p,q \le \infty$, then

$$\|f * g\|_{L^r(\Gamma)} \le \|f\|_{L^p(\Gamma)} \|g\|_{L^q(\Gamma)}, \quad \text{where } \frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \tag{1.17}$$

and

$$(f*g)(\mathbf{x}) := \int_\Gamma f(\mathbf{x}-\mathbf{y}) g(\mathbf{y})\,\mathrm{d}s(\mathbf{y}).$$

Young's inequality is usually stated for Lebesgue spaces defined on $\mathbb{R}^d$, however the proof only depends on Hölder's inequality, and since the latter inequality holds for Lebesgue spaces defined on Γ, so does the former.

The function $S_k\phi(\mathbf{x})$ is a convolution (since $\Phi_k(\mathbf{x},\mathbf{y})$ is a function of $\mathbf{x}-\mathbf{y}$), and applying Young's inequality (1.17) with $p=1$, $q=2$, $r=2$, and $g=\phi$ yields

$$\|S_k\phi\|_{L^2(\Gamma)} \le \left(\int_\Gamma \left|f(\mathbf{z})\right| \mathrm{d}s(\mathbf{z})\right) \|\phi\|_{L^2(\Gamma)}, \tag{1.18}$$

where $f(\mathbf{z}) := \mathrm{i}H_0^{(1)}(k|\mathbf{z}|)/4$ for $d=2$ and $\exp(\mathrm{i}k|\mathbf{z}|)/(4\pi|\mathbf{z}|)$ for $d=3$ (compare (1.18) to (1.12)). Using the bounds (1.13) (for $d=2$) and (1.15) (for $d=3$) in (1.18) then yields the bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in (1.9).

Neither of the functions $D_k\phi(\mathbf{x})$ and $(D_k - D_0)\phi(\mathbf{x})$ are convolutions (since their kernels contain $\mathbf{n}(\mathbf{y})$ which is not a function of $\mathbf{x}-\mathbf{y}$), and thus Young's inequality cannot immediately be applied. Nevertheless, the kernels of both D_k and $D_k - D_0$ can be bounded by functions of $\mathbf{x}-\mathbf{y}$, and then Young's inequality can in principle be used to bound $D_k\phi(\mathbf{x})$ and $(D_k - D_0)\phi(\mathbf{x})$. This procedure yields no information when applied to D_k, since its kernel has too strong a singularity to be bounded in L^1, however applying this procedure to $D_k - D_0$ and using bounds analogous to (1.13) and (1.15) (see [6, Equation 3.9 and Lemma 3.4]) yields the bounds on $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ given in (1.9).

1.3 Motivation for the current investigation: coercivity of $A'_{k,\eta}$ and $A_{k,\eta}$

Given a BVP for the Helmholtz equation, its standard variational formulation (i.e. the weak form of the BVP) satisfies a Gårding inequality, and thus the operator associated with this variational formulation is a compact perturbation of a coercive operator. Furthermore, one can prove that, even when the BVP has a unique solution for all k, the standard variational formulation is not coercive (see, e.g., [20, §1.1]), and thus coercivity up to a compact perturbation is the best result one can obtain for this formulation.

The standard analysis of boundary integral operators (BIOs) in a variational setting "transfers" the coercivity properties of the weak form of the BVP to the relevant BIOs posed in the trace spaces. This method therefore proves that the standard first- and second-kind integral operators used to solve the Helmholtz equation are compact perturbations of coercive operators (see [10] and [24, §1.4] for overviews of this method).

Despite the fact that the standard variational formulations are not coercive, there do exist coercive variational formulations of Helmholtz BVPs (these are summarised in [20, §1.2]). In particular, the combined potential operators $A'_{k,\eta}$ and $A_{k,\eta}$, defined by (1.4) and (1.6) respectively, were proved to be coercive when Γ is the circle or sphere, $\eta = k$, and k is sufficiently large in [11, Theorems 4.2 and 4.12], and numerical experiments in [4] suggest that these operators are coercive whenever Ω_+ is nontrapping, $\eta = k$, and k is sufficiently large.

The main result of this paper (Theorem 1.4 below) is a component of the proof of the following theorem, which enlarges the class of domains for which $A'_{k,\eta}$ and $A_{k,\eta}$ are proved to be coercive.

Theorem 1.3 [24, Theorem 1.2] *Let Ω_- be 2- or 3-d domain whose boundary, Γ, has strictly positive curvature and is both C^3 and piecewise analytic. Then there exists a constant $\eta_0 > 0$ such that, given $\delta > 0$, there exists $k_0 > 0$ (depending on δ) such that, for $k \geq k_0$ and $\eta \geq \eta_0 k$,*

$$\Re\big(A'_{k,\eta}\phi,\phi\big)_{L^2(\Gamma)} \geq \left(\frac{1}{2}-\delta\right)\|\phi\|^2_{L^2(\Gamma)} \tag{1.19}$$

for all $\phi \in L^2(\Gamma)$ (where, for $z \in \mathbb{C}$, $\Re z$ denotes the real part of z).

(Note that, by the relations in (1.7), the bound (1.19) also holds when the direct integral operator $A'_{k,\eta}$ is replaced by the indirect operator $A_{k,\eta}$.)

To see how bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ are needed in the proof of this result, note that in [24] it is shown that if Ω_- satisfies the conditions in the theorem then there exists a constant η_0 and a function α (both independent of k) such that, if $\eta \geq \eta_0 k$ and k is sufficiently large,

$$\Re\big(A'_{k,\eta}\phi,\phi\big)_{L^2(\Gamma)} + \big(\alpha S_k\phi,\phi\big)_{L^2(\Gamma)} \geq \frac{1}{2}\|\phi\|^2_{L^2(\Gamma)} \tag{1.20}$$

for all $\phi \in L^2(\Gamma)$ (see [24, Equation 3.12 onwards] and note that the α in (1.20) is equal to α/C_Γ in [24]). Using the Cauchy–Schwarz inequality in (1.20), we find that

$$\Re\big(A'_{k,\eta}\phi,\phi\big)_{L^2(\Gamma)} \geq \left(\frac{1}{2} - \|\alpha\|_{L^\infty(\Gamma)}\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\right)\|\phi\|^2_{L^2(\Gamma)}. \tag{1.21}$$

Therefore, if $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \to 0$ as $k \to \infty$ then the inequality (1.21) shows that $A'_{k,\eta}$ is coercive (for these Ω_-) when k is sufficiently large.

When $d = 2$, the bound (1.14) shows that $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \to 0$ as $k \to \infty$. However, the corresponding bound when $d = 3$, namely (1.16), does not give this required decay. (When Γ is a sphere this decay is ensured by (1.8), although coercivity of $A'_{k,\eta}$ in this case can be established by bounding the eigenvalues of $A'_{k,\eta}$; see [11, Theorem 4.12], [7, §5.4].)

The main goal of this paper is to prove that if Ω_- satisfies the geometric assumptions in Theorem 1.3 (or less restrictive assumptions) then $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \to 0$ as $k \to \infty$, and this is achieved in Theorem 1.4 below.

1.4 The main result of this paper

In this paper we prove an upper bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ when Ω_- is a 3–d, C^2 domain with strictly positive curvature (motivated by the need to prove that $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \to 0$ as $k\to\infty$ on these domains discussed in §1.3 above). We emphasise, however, that the method we use to prove this bound can also be used to bound $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ on this class of domains, and can also, in principle, be used to bound $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ on more general domains in both 2– and 3–d; we discuss this more in Remark 1.1 below.

Before we state the main result we recall some facts about curvature. Assume that $\Omega_-\subset\mathbb{R}^3$ and Γ is C^2. Recall that the two principal curvatures at a point $\mathbf{x}\in\Gamma$ are the maximum and minimum of the curvatures at $\mathbf{x}$ of all the 1–d curves on Γ passing through $\mathbf{x}$. We need to choose a sign-convention when dealing with the curvature of 1–d curves; we choose the sign so that a circle has positive curvature. We say that Γ has *strictly positive curvature* if there exists a $\kappa_0>0$ such that, for any $\mathbf{x}\in\Gamma$, the principal curvatures at $\mathbf{x}$ are both $\geq\kappa_0$. Note that our sign-convention for curvature implies that if Γ has strictly positive curvature then Ω_- is strictly convex (but the converse is not true).

Theorem 1.4 *Let Ω_- be a 3–d domain whose boundary, Γ, is C^2 and has strictly positive curvature. Then, given $k_0>1$, there exists a $C>0$ such that*

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\frac{(\log k)^{1/2}}{k^{1/13}} \tag{1.22}$$

for all $k\geq k_0$.

This bound should be compared with the only other existing bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ for this type of domain, namely $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$ (1.16).

Note that to prove the bound (1.22) we only need to show that there exists a $k_1>1$ and $C'>0$ such that

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C'\frac{(\log k)^{1/2}}{k^{1/13}} \quad \text{for all } k\geq k_1. \tag{1.23}$$

Indeed, if we have shown that (1.23) holds then, given any $k_0>1$, we define C by

$$C := \max\left\{C', \left[\min_{k_0\leq k\leq k_1}\left(\frac{(\log k)^{1/2}}{k^{1/13}}\right)\right]^{-1}\left[\max_{k_0\leq k\leq k_1}\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\right]\right\}$$

(where $\max_{k_0\leq k\leq k_1}\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}<\infty$ since S_k is a bounded operator on $L^2(\Gamma)$ for every $k>0$), and then (1.22) holds with this particular value of C.

1.5 Overview of the proof of Theorem 1.4

The main idea is to use the fact that

$$\|S_k\|^2_{L^2(\Gamma)\to L^2(\Gamma)} = \|S_k^*S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \tag{1.24}$$

(where S_k^* is the Hilbert-space adjoint of the operator S_k on $L^2(\Gamma)$), and then use the Riesz-Thorin method explained in §1.2 to bound $\|S_k^*S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

This idea of bounding the L^2–norm of an oscillatory integral operator T by bounding the L^2–norm of T^*T is well known in the harmonic analysis literature, see, e.g., [25, Page 279], and its use in the context of integral operators associated with the Helmholtz equation was first suggested in [5, Page 184].

Using the fact that

$$S_k^*\phi(\mathbf{x}) = \int_\Gamma \overline{\Phi(\mathbf{y},\mathbf{x})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

we have that

$$S_k^*S_k\phi(\mathbf{x}) = \int_\Gamma t_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

where

$$t_k(\mathbf{x},\mathbf{y}) := \int_\Gamma \overline{\Phi_k(\mathbf{z},\mathbf{x})}\,\Phi_k(\mathbf{z},\mathbf{y})\,\mathrm{d}s(\mathbf{z}) = \frac{1}{16\pi^2}\int_\Gamma \frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}). \tag{1.25}$$

(Interchanging the order of integration in $S_k^* S_k$ can be justified using Fubini's theorem and Tonelli's theorem, since each of the iterated integrals converges absolutely; see [12, Remark (iv) after Theorem 2.37].)

We can then use the Riesz–Thorin method outlined in §1.2 to obtain

$$\|S_k^* S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \left(\operatorname*{ess\,sup}_{\mathbf{y}\in\Gamma}\int_\Gamma \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{x})\right)^{1/2}\left(\operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma}\int_\Gamma \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\right)^{1/2}$$

and then, since $\overline{t_k(\mathbf{x},\mathbf{y})} = t_k(\mathbf{y},\mathbf{x})$, we have that

$$\|S_k^* S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma}\int_\Gamma \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}). \tag{1.26}$$

We saw in §1.2 that both the Riesz–Thorin method and the method using Young's inequality can be used to obtain the bounds (1.9). Only the Riesz–Thorin method, however, is applicable when bounding $\|S_k^* S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, since the function $S_k^* S_k\phi(\mathbf{x})$ is not a convolution (as $t_k(\mathbf{x},\mathbf{y})$ is not a function of $\mathbf{x}-\mathbf{y}$).

The steps above reduce bounding $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ to bounding the kernel $t_k(\mathbf{x},\mathbf{y})$ defined by (1.25). We now outline the main steps in this argument.

Bounding the kernel $t_k(\mathbf{x},\mathbf{y})$. The first thing to note is that, when $\mathbf{x}=\mathbf{y}$, the integral in (1.25) is strongly singular, and thus $|t_k(\mathbf{x},\mathbf{x})|$ is infinite. Our plan, therefore, is to choose an arbitrary $\mathbf{x}\in\Gamma$, fix $\varepsilon>0$, and split the range of integration into $\Gamma\cap B_\varepsilon(\mathbf{x})$ and $\Gamma\setminus B_\varepsilon(\mathbf{x})$ (where $B_\varepsilon(\mathbf{x}) := \{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}|<\varepsilon\}$). Then

$$\int_\Gamma \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}) = J_1(\mathbf{x};k,\varepsilon) + J_2(\mathbf{x};k,\varepsilon),$$

where

$$J_1(\mathbf{x};k,\varepsilon) := \int_{\Gamma\cap B_\varepsilon(\mathbf{x})} \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}) \quad \text{and} \quad J_2(\mathbf{x};k,\varepsilon) := \int_{\Gamma\setminus B_\varepsilon(\mathbf{x})} \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}).$$

We now proceed to bound J_1 and J_2 separately. Note that all the resulting bounds are obtained assuming that ε is sufficiently small (and we do not state this explicitly every time).

Overview of the bound on J_1. Our plan for J_1 is to bound $|t_k(\mathbf{x},\mathbf{y})|$ explicitly in terms of $|\mathbf{x}-\mathbf{y}|$, k, and ε for $\mathbf{y}\in\Gamma\cap B_\varepsilon(\mathbf{x})$, and then integrate this bound, i.e. we find $b(|\mathbf{x}-\mathbf{y}|,k,\varepsilon)$ such that

$$\big|t_k(\mathbf{x},\mathbf{y})\big| \lesssim b\big(|\mathbf{x}-\mathbf{y}|,k,\varepsilon\big) \quad \text{for } \mathbf{y}\in\Gamma\cap B_\varepsilon(\mathbf{x}) \tag{1.27}$$

(where b is given explicitly in terms of $|\mathbf{x}-\mathbf{y}|$, k, and ε), and then use

$$\int_{\Gamma\cap B_\varepsilon(\mathbf{x})} \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}) \lesssim \int_{\Gamma\cap B_\varepsilon(\mathbf{x})} b\big(|\mathbf{x}-\mathbf{y}|,k,\varepsilon\big)\,\mathrm{d}s(\mathbf{y}).$$

We do this in §3, and find that we can take the function b to be

$$b\big(|\mathbf{x}-\mathbf{y}|,k,\varepsilon\big) = \log\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) + 1, \tag{1.28}$$

and we therefore obtain that

$$J_1(\mathbf{x};k,\varepsilon) = \int_{\Gamma\cap B_\varepsilon(\mathbf{x})} \big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y}) \lesssim \varepsilon^2\log\left(\frac{1}{\varepsilon}\right). \tag{1.29}$$

It is instructive to observe if $t_k(\mathbf{x},\mathbf{y})$ were bounded when $\mathbf{x}=\mathbf{y}$ then we would obtain the bound $J_1(\mathbf{x};k,\varepsilon)\lesssim \varepsilon^2$, and thus the bound (1.29) is (in some sense) almost optimal.

One novelty of the bound (1.28) is that, to obtain it, we use techniques that determine the asymptotics of integrals with algebraic parameter dependence, and these techniques are perhaps not so well known, even in the asymptotics literature. Indeed, whereas techniques for obtaining the asymptotics of integrals with *exponential* parameter dependence (e.g. Watson's lemma, the method of stationary phase, and the method of steepest descent) appear in many books, to the author's knowledge, the only book that describes the techniques for integrals with *algebraic* parameter dependence is [17]. (For the reader specifically interested in our use of these techniques, this can be found in Appendix A.)

Overview of the bound on J_2. Our plan for J_2 is to bound $|t_k(\mathbf{x},\mathbf{y})|$ explicitly in terms of $|\mathbf{x}-\mathbf{y}|$, k, and ε for $\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})$ and then use the inequality

$$\int_{\Gamma\setminus B_\varepsilon(\mathbf{x})}\big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\leq|\Gamma|\max_{\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})}\big|t_k(\mathbf{x},\mathbf{y})\big|, \tag{1.30}$$

where $|\Gamma|$ denotes the surface area of Γ. Since $|\mathbf{x}-\mathbf{y}|\geq\varepsilon$, $t_k(\mathbf{x},\mathbf{y})$ is always finite, but the kernel of the integral defining $t_k(\mathbf{x},\mathbf{y})$ is weakly singular at both $\mathbf{z}=\mathbf{x}$ and $\mathbf{z}=\mathbf{y}$.

The argument we use to obtain a bound on $t_k(\mathbf{x},\mathbf{y})$ when $\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})$ is quite technical (indeed, the integral defining $t_k(\mathbf{x},\mathbf{y})$ is split into the sum of 6 separate integrals) however the guiding philosophy is to localise near weak singularities with cut-off functions until one obtains an (oscillatory) integral with no weak singularities that can be integrated by parts.

The end result is the bound

$$J_2(\mathbf{x};k,\varepsilon)=\int_{\Gamma\setminus B_\varepsilon(\mathbf{x})}\big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\lesssim\varepsilon^{n-1}+\frac{1}{k\,\varepsilon^{2n+5}}. \tag{1.31}$$

for any $n>1$.

Obtaining the final result (1.22) (by "gearing" ε to k). Combining (1.29) and (1.31) we have that, for any $\mathbf{x}\in\Gamma$,

$$\int_\Gamma\big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\lesssim\varepsilon^2\log\left(\frac{1}{\varepsilon}\right)+\varepsilon^{n-1}+\frac{1}{k\varepsilon^{2n+5}} \tag{1.32}$$

We now choose ε to make the last two terms on the right-hand side of (1.32) the same order of magnitude. Since

$$\varepsilon^a\sim\frac{1}{k\,\varepsilon^b}\quad\text{when}\quad\varepsilon\sim\frac{1}{k^{1/(a+b)}},$$

we choose

$$\varepsilon=\frac{1}{k^{1/(3n+4)}},$$

and then (1.32) becomes

$$\int_\Gamma\big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\lesssim\frac{\log k}{k^{2/(3n+4)}}+\frac{1}{k^{(n-1)/(3n+4)}} \tag{1.33}$$

(we can assume that $\varepsilon<1$ so that $k>1$ and thus $\log k>0$).

For the first term on the right-hand side of (1.33) to be small we want n to be small, but for the second term to be small we want n to be large (since

$$\frac{n-1}{3n+4}=\frac{1}{3}\left(1-\frac{7}{3n+4}\right)$$

and we want this to be as large as possible). The optimal value of n is therefore the value for which the powers of k in the two terms are equal, and this is 3. Substituting $n=3$ into (1.33) and using (1.26) we obtain

$$\|S_k^*S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim\sup_{\mathbf{x}\in\Gamma}\int_\Gamma\big|t_k(\mathbf{x},\mathbf{y})\big|\,\mathrm{d}s(\mathbf{y})\lesssim\frac{\log k}{k^{2/13}}.$$

Therefore, using (1.24), we have that there exists a $k_1>1$ such that

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim\frac{(\log k)^{1/2}}{k^{1/13}}\quad\text{for all }k\geq k_1;$$

this is the bound (1.23), from which the result (1.22) follows.

Integration by parts. Although we have skipped over most of the details of the proof in this overview section, it is instructive to give a few of the details of the integration by parts that occurs when estimating J_2 (since these then motivate our investigation of the integral's phase function in §2 below).

After removing the weak singularities from the integrand using cut-off functions, we arrive in our estimation of J_2 at the integral

$$\int_\Gamma e^{ik\phi(\mathbf{z};\mathbf{x},\mathbf{y})} f(\mathbf{z};\mathbf{x},\mathbf{y})\,ds(\mathbf{z}), \tag{1.34}$$

where the phase function ϕ is given by

$$\phi(\mathbf{z};\mathbf{x},\mathbf{y}) := |\mathbf{z}-\mathbf{y}| - |\mathbf{z}-\mathbf{x}|, \tag{1.35}$$

and the integrand f is given by

$$f(\mathbf{z};\mathbf{x},\mathbf{y}) = \Big(1-\chi_{\delta,\mathbf{y}}(\mathbf{z})\Big)\Big(1-\chi_{\delta,\mathbf{x}}(\mathbf{z})\Big)\frac{1}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}. \tag{1.36}$$

The function $\chi_{\delta,\mathbf{y}}(\mathbf{z})$ equals one in $B_\delta(\mathbf{y})$ and zero outside $B_{2\delta}(\mathbf{y})$ (see (3.1) below). The function $f(\mathbf{z};\mathbf{x},\mathbf{y})$ is therefore zero when $\mathbf{z}\in B_\delta(\mathbf{y})$ and $\mathbf{z}\in B_\delta(\mathbf{x})$, and the range of integration in (1.34) can then be changed from Γ to $\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$. (Note that in the proof we also take $\delta\ll\varepsilon$ so that when $\mathbf{y}\notin B_\varepsilon(\mathbf{x})$, we have that $\mathbf{y}\notin B_\delta(\mathbf{x})$.)

The integral (1.34) is an oscillatory integral with no weak singularities, and we can therefore integrate by parts (to condense notation, we suppress the dependence of ϕ and f on $\mathbf{x}$ and $\mathbf{y}$ from now on in this section). We begin by observing that

$$e^{ik\phi(\mathbf{z})} f(\mathbf{z}) = \frac{1}{ik}\nabla_\Gamma\left(e^{ik\phi(\mathbf{z})}\right)\cdot\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2} f(\mathbf{z}), \tag{1.37}$$

where ∇_Γ is the surface gradient (defined in §1.6 below). This equation shows us that we need to know if $\nabla_\Gamma\phi(\mathbf{z})$ can be zero (i.e. if the integral has any stationary points).

The definition of $\phi(\mathbf{z})$ (1.35) implies that it is a differentiable function of $\mathbf{z}$ when $\mathbf{z}$ is not equal to $\mathbf{x}$ or $\mathbf{y}$. Furthermore, Lemma 2.1 below shows that if Ω_- is strictly convex, $\mathbf{x}\neq\mathbf{y}$, and $\mathbf{z}$ is not equal to either $\mathbf{y}$ or $\mathbf{x}$, then $\nabla_\Gamma\phi(\mathbf{z})$ is never zero.

Therefore, when Ω_- is strictly convex we can use (1.37) in (1.34) and obtain

$$\int_{\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))} e^{ik\phi(\mathbf{z})} f(\mathbf{z})\,ds(\mathbf{z}) = \frac{1}{ik}\int_{\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))}\nabla_\Gamma\left(e^{ik\phi(\mathbf{z})}\right)\cdot\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2} f(\mathbf{z})\,ds(\mathbf{z}). \tag{1.38}$$

We can now use the divergence theorem (or equivalently Stokes' theorem) to move the ∇_Γ from the exponential in the integrand of the right-hand side of (1.38) onto the other terms in the integrand; bounding the resulting integral (in combination with all the omitted steps) leads to the bound on J_2 (1.31).

Before concluding this discussion we make two remarks: (i) The equation (1.38) shows that we need a lower bound on $|\nabla_\Gamma\phi|$ for $\mathbf{z}\in\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$ and $\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})$ (recall that this second condition comes from the fact that we are estimating J_2), and we need this bound to be valid as ε and $\delta\to 0$ (with $\delta\ll\varepsilon$). (ii) Integrating by parts the right-hand side of (1.38) requires $\nabla_\Gamma\phi$ to be differentiable, and this requires that Γ be C^2 (since differentiating the surface gradient at $\mathbf{x}\in\Gamma$ requires differentiating the tangent vectors to Γ at $\mathbf{x}$). If we assume that Γ is smoother than C^2 then further integration by parts are allowed (if Γ is C^m, then we can integrate by parts $m-1$ times), but we do not do this here.

Remark 1.1 (Bounds for $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and for more general domains) When Γ is C^2, the singularity in the kernel of D_k is the same as that in S_k. It should not be too difficult, therefore, to adapt the argument leading to Theorem 1.4 to prove an analogous result for $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ on this type of domain. It should also not be too difficult to translate the arguments for S_k and D_k when Ω_- is a 3–d, C^2 domain with strictly positive curvature to the case when Ω_- is a 2–d, C^2 domain with strictly positive curvature (although a complicating factor in 2–d is that one requires appropriate bounds on the Hankel function $H_0^{(1)}$).

In principle, this type of argument could be applied to domains that are not strictly convex. However, the presence of stationary points of the phase function ϕ would then make the argument for these domains much more complicated than that for strictly convex domains.

Remark 1.2 (The results of [13]) Whilst this paper was being written, Galkowski and Smith [13] also investigated the wavenumber-dependence of $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$. By using restriction estimates for eigenfunctions of the Laplacian, these authors proved sharper bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ than the bound in Theorem 1.4. Indeed, [13, Theorem 2] (see also [16, Appendix A]) states that if $d = 2$ or 3 and Γ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces then, given $k_0 > 1$,

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \frac{\log k}{k^{1/2}}, \tag{1.39}$$

for all $k \geq k_0$, and if Γ is a finite union of compact subsets of strictly convex $C^{2,1}$ hypersurfaces then, given $k_0 > 1$

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \frac{\log k}{k^{2/3}} \tag{1.40}$$

for all $k \geq k_0$. Observe that both the bounds (1.40) and (1.22) hold when Γ is $C^{2,1}$ and has strictly positive curvature, but that (1.40) is stronger. Furthermore, [16, §A.2] shows that the powers of k in (1.39) and (1.40) are optimal (i.e. the bounds are sharp modulo the log loss).

1.6 Notation and basic results

Notation for asymptotics.

- $a = o(b)$ as $\varepsilon \to 0$ means that
$$\frac{a}{b} \to 0 \quad \text{as } \varepsilon \to 0.$$
- $a = \mathscr{O}(b)$ as $\varepsilon \to 0$ means that there exists an $\varepsilon_0 > 0$ and $C > 0$ such that
$$\left|\frac{a}{b}\right| \leq C \quad \text{for all } \varepsilon \geq \varepsilon_0$$
(where the constant C is independent of ε).
- $a \ll b$ as $\varepsilon \to 0$ means $a = o(b)$ as $\varepsilon \to 0$, and $a \gg b$ as $\varepsilon \to 0$ means $b \ll a$ as $\varepsilon \to 0$. (The advantage of using this notation in addition to the little-o notation is that we do not need to write expressions such as $\varepsilon = o(|\mathbf{z}-\mathbf{x}|)$, but can write $|\mathbf{z}-\mathbf{x}| \gg \varepsilon$ instead.)
- $a \sim b$ means $a = \mathscr{O}(b)$ and $b = \mathscr{O}(a)$ as $\varepsilon \to 0$ (another commonly used notation for this is $a = \mathrm{ord}(b)$, but we do not use this here). Note that this differs from the standard definition that $a \sim b$ iff $a/b \to 1$ as $\varepsilon \to 0$.
- $a \lesssim b$ means $a \leq Cb$ where C is independent of ε. (Observe that $a \lesssim b$ implies that $a = \mathscr{O}(b)$, but $a = \mathscr{O}(b)$ implies that $|a| \lesssim |b|$.) The notation $a \gtrsim b$ means that $b \lesssim a$.

Differential operators on a surface. We restrict attention to the case that Γ is C^2 (although the results we recall below hold when Γ is Lipschitz after some extra technical work).

The surface gradient, ∇_Γ, is defined in terms of a parametrisation of the boundary in, e.g., [8, §2.1], [23, Equation 2.5.176], [7, Equation A.14]. Recall that if u is differentiable in a neighbourhood of Γ, then

$$\nabla u(\mathbf{x}) = \nabla_\Gamma u(\mathbf{x}) + \mathbf{n}(\mathbf{x})\frac{\partial u}{\partial n}(\mathbf{x}) \quad \text{when } \mathbf{x} \in \Gamma.$$

The surface divergence, $\nabla_\Gamma \cdot$, is defined in terms of a parametrisation of the boundary in, e.g., [23, Equation 2.5.205], [21, §3.4]. Later we use the fact that ∇_Γ and $\nabla_\Gamma \cdot$ are such that the identity

$$\nabla_\Gamma \cdot (\phi \mathbf{F}) = \nabla_\Gamma \phi \cdot \mathbf{F} + \phi\, \nabla_\Gamma \cdot \mathbf{F} \tag{1.41}$$

holds when ϕ is a scalar field and $\mathbf{F}$ is a vector field tangent to Γ.

Let $S \subset \mathbb{R}^3$ be such that $\overline{S}$ is a compact, 2–d, C^2 submanifold with boundary of $\mathbb{R}^3$, and assume that S and ∂S are both locally the graphs of functions. Let $\boldsymbol{\nu}$ be one of the two unit normal vectors to S and, having chosen $\boldsymbol{\nu}$, let $\boldsymbol{\tau}$ be the unit tangent vector to ∂S such that $\boldsymbol{\tau}$ points anti-clockwise when $\boldsymbol{\nu}$ points towards the observer. (Later we take S to be a subset of Γ, in which case we take $\boldsymbol{\nu}$ to be $\mathbf{n}$.)

Lemma 1.1 (The divergence theorem on surfaces) *With S, $\boldsymbol{\nu}$, and $\boldsymbol{\tau}$ as above, if* $\mathbf{F} \in (C^1(\overline{S}))^3$, *then*

$$\int_S \nabla_S \cdot \mathbf{F} \, \mathrm{d}s = \int_{\partial S} \mathbf{F} \cdot (\boldsymbol{\tau} \times \boldsymbol{\nu}) \, \mathrm{d}\sigma,$$

where ∇_S *is the surface gradient on S,* $\mathrm{d}s$ *is the 2–d surface measure on S, and* $\mathrm{d}\sigma$ *is the 1–d measure on the curve* ∂S.

Proof This result follows from Stokes' theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot \boldsymbol{\nu} \, \mathrm{d}s = \int_{\partial S} \mathbf{A} \cdot \boldsymbol{\tau} \, \mathrm{d}\sigma,$$

with $\mathbf{A} = \boldsymbol{\nu} \times \mathbf{F}$, noting that (i) $(\nabla \times \mathbf{A}) \cdot \boldsymbol{\nu} = \nabla_S \cdot (\mathbf{A} \times \boldsymbol{\nu})$ [9, Equation 6.38], and (ii) $\nabla_S \cdot \big((\boldsymbol{\nu} \times \mathbf{F}) \times \boldsymbol{\nu}\big) = \nabla_S \cdot \mathbf{F}$ on S (since $(\boldsymbol{\nu} \times \mathbf{F}) \times \boldsymbol{\nu}$ is the tangential component of the restriction of $\mathbf{F}$ to S, and $\nabla_S \cdot$ acts on the space of vector fields tangent to the surface S).

2 Understanding the behaviour of the phase function $\phi(\mathbf{z};\mathbf{x},\mathbf{y})$

Although the main result of this paper concerns the case when Ω_- is 3–d, in this section we consider both 2–d and 3–d Ω_-. This is because many of the arguments in the 3–d case can be reduced to their 2–d counterparts.

We first show that if Ω_- is strictly convex and $\mathbf{x} \neq \mathbf{y}$, then the phase function $\phi(\mathbf{z};\mathbf{x},\mathbf{y})$ defined by (1.35) has no stationary points.

Lemma 2.1 *Let* $\Omega_- \subset \mathbb{R}^d$, *with* $d = 2$ *or* 3, *be strictly convex. If* $\mathbf{z} \neq \mathbf{x}$, $\mathbf{z} \neq \mathbf{y}$, *and* $\mathbf{x} \neq \mathbf{y}$, *then* $\nabla_\Gamma \phi(\mathbf{z};\mathbf{x},\mathbf{y}) \neq \mathbf{0}$.

Proof We first show that $\nabla \phi(\mathbf{z};\mathbf{x},\mathbf{y}) \neq \mathbf{0}$. (In this paper all derivatives of ϕ are with respect to the $\mathbf{z}$ variable, and so we do not write this explicitly.)

When $\mathbf{z} \neq \mathbf{x}$ or $\mathbf{y}$, ϕ is a differentiable function of $\mathbf{z}$ and

$$\nabla \phi(\mathbf{z};\mathbf{x},\mathbf{y}) = \frac{\mathbf{z}-\mathbf{y}}{|\mathbf{z}-\mathbf{y}|} - \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|} = \widehat{\mathbf{z}-\mathbf{y}} - \widehat{\mathbf{z}-\mathbf{x}}. \tag{2.1}$$

Seeking a contradiction, we assume that there exists a $\mathbf{z}^* \in \Gamma$ such that $\nabla \phi(\mathbf{z}^*;\mathbf{x},\mathbf{y}) = \mathbf{0}$. Then $\widehat{\mathbf{z}^*-\mathbf{y}} = \widehat{\mathbf{z}^*-\mathbf{x}}$ and so $\mathbf{z}^*$, $\mathbf{x}$, and $\mathbf{y}$ must be collinear (i.e. lie on a single straight line). Since Ω_- is strictly convex, this cannot happen.

We next show that $\nabla_\Gamma \phi(\mathbf{z};\mathbf{x},\mathbf{y}) \neq \mathbf{0}$. Again seeking a contradiction, we assume that there exists a $\mathbf{z}^* \in \Gamma$ such that $\nabla_\Gamma \phi(\mathbf{z}^*;\mathbf{x},\mathbf{y}) = \mathbf{0}$. The expression for $\nabla \phi(\mathbf{z}^*;\mathbf{x},\mathbf{y})$ (2.1) implies that $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ must have equal tangential components. Since $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ are both unit vectors there are then three possibilities,

1. the normal components of $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ are equal and nonzero,
2. the normal components of $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ are both equal to zero, and
3. the normal components of $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ are equal in modulus but have opposite sign.

If 1 or 2 held then $\widehat{\mathbf{z}^*-\mathbf{y}}$ and $\widehat{\mathbf{z}^*-\mathbf{x}}$ would be equal and then $\nabla \phi(\mathbf{z}^*;\mathbf{x},\mathbf{y})$ would be equal to zero, but this cannot happen by the argument above. By strict convexity $\mathbf{n}(\mathbf{z}^*) \cdot (\widehat{\mathbf{z}^*-\mathbf{x}}) > 0$ and $\mathbf{n}(\mathbf{z}^*) \cdot (\widehat{\mathbf{z}^*-\mathbf{y}}) > 0$, thus 3 cannot hold either.

In the proof of the main result (Theorem 1.4) it turns out that we need a lower bound on $|\nabla_\Gamma \phi|$ when $\mathbf{z} \in \Gamma \setminus (B_\delta(\mathbf{x}) \cup B_\delta(\mathbf{y}))$ and $|\mathbf{x}-\mathbf{y}| \gtrsim \varepsilon$, with the parameters ε and δ allowed to be arbitrarily small (one can see this from (1.38), since we need to bound the integral on the right-hand side of this equation).

Lemma 2.2 (A lower bound on $\nabla_\Gamma \phi$) *Let* $\Omega_- \subset \mathbb{R}^d$, *with* $d = 2$ *or* 3, *be such that* Γ *is* C^2 *and has strictly positive curvature. If* $\mathbf{x},\mathbf{y} \in \Gamma$ *with* $|\mathbf{x}-\mathbf{y}| \gtrsim \varepsilon$, *and* $\mathbf{z} \in \Gamma \setminus (B_\delta(\mathbf{x}) \cup B_\delta(\mathbf{y}))$, *then*

$$|\nabla_\Gamma \phi(\mathbf{z};\mathbf{x},\mathbf{y})| \gtrsim \varepsilon^2 \tag{2.2}$$

as ε *and* $\delta \to 0$ *with* $\delta \ll \varepsilon$ *(where the omitted constant in (2.2) is independent of* $\mathbf{z},\mathbf{x},\mathbf{y},\varepsilon$, *and* δ*).*

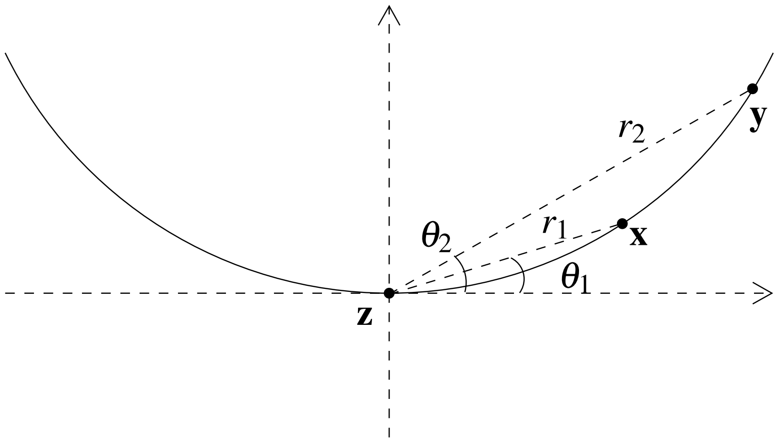

Fig. 2.1 The surface Γ (in 2–d) in a neighbourhood of **z**

Proof We split the proof up into 3 cases:

Case (i) $|\mathbf{x}-\mathbf{y}| \sim \varepsilon$.
Case (ii) $\varepsilon \ll |\mathbf{x}-\mathbf{y}| \ll 1$.
Case (iii) $|\mathbf{x}-\mathbf{y}| \sim 1$.

In Case (iii) we claim that $|\nabla_\Gamma \phi| \gtrsim 1$ as ε and $\delta \to 0$. More precisely, we claim that given $C_1 > 0$ there exists a C_2 (dependent on C_1 but independent of ε and δ) such that given any $\delta > 0$, if $\mathbf{z} \in \Gamma \setminus (B_\delta(\mathbf{x}) \cup B_\delta(\mathbf{y}))$ and $|\mathbf{x}-\mathbf{y}| \geq C_1$ then $|\nabla_\Gamma \phi(\mathbf{z};\mathbf{x},\mathbf{y})| \geq C_2$. Indeed, seeking a contradiction, we suppose that this is not true. Then there exists a $C_1 > 0$ and $(\mathbf{z}_n,\mathbf{x}_n,\mathbf{y}_n,\delta_n)_{n=1}^\infty$ such that $\delta_n > 0$, $|\mathbf{z}_n-\mathbf{x}_n| \geq \delta_n$, $|\mathbf{z}_n-\mathbf{y}_n| \geq \delta_n$, $|\mathbf{x}_n-\mathbf{y}_n| \geq C_1$ and $|\nabla_\Gamma \phi(\mathbf{z}_n;\mathbf{x}_n,\mathbf{y}_n)| \leq 1/n$. Since $\nabla_\Gamma \phi(\mathbf{z}_n;\mathbf{x}_n,\mathbf{y}_n) \to 0$, the tangential component of $\widehat{\mathbf{z}_n-\mathbf{y}_n}$ must tend to the tangential component of $\widehat{\mathbf{z}_n-\mathbf{x}_n}$ (see (2.1)). However, this is impossible since $\mathbf{x}_n \not\to \mathbf{y}_n$ and Ω_- is strictly convex.

We now consider Case (i) (and it turns out that after we have proved the bound for Case (i), the bound for Case (ii) follows immediately).

Case (i) Without loss of generality, we assume that $|\mathbf{x}-\mathbf{y}| = \varepsilon$ and $|\mathbf{z}-\mathbf{x}| \lesssim |\mathbf{z}-\mathbf{y}|$. We can then divide the set $\{\mathbf{z} : \mathbf{z} \in \Gamma \setminus (B_\delta(\mathbf{x}) \cup B_\delta(\mathbf{y}))\}$ into the following 5 regimes:

Regime 1. $|\mathbf{z}-\mathbf{x}| \sim \delta$ (so $|\mathbf{z}-\mathbf{y}| \sim \varepsilon$).
Regime 2. $\delta \ll |\mathbf{z}-\mathbf{x}| \ll \varepsilon$ (so $|\mathbf{z}-\mathbf{y}| \sim \varepsilon$).
Regime 3. $|\mathbf{z}-\mathbf{x}| \sim \varepsilon$ (so $|\mathbf{z}-\mathbf{y}| \sim \varepsilon$).
Regime 4. $\varepsilon \ll |\mathbf{z}-\mathbf{x}| \ll 1$ (so $|\mathbf{z}-\mathbf{y}| \sim |\mathbf{z}-\mathbf{x}|$).
Regime 5. $|\mathbf{z}-\mathbf{x}| \sim 1$ (so $|\mathbf{z}-\mathbf{y}| \sim |\mathbf{z}-\mathbf{x}|$).

The case when Ω_- is 2–d. Introduce polar co-ordinates (r,θ) with origin at $\mathbf{z} \in \Gamma$ and with the horizontal axis (corresponding to $\theta = 0$) tangent to Γ at $\mathbf{z}$; see Figure 2.1.

Let $\mathbf{x}-\mathbf{z}$ correspond to (r_1,θ_1) and $\mathbf{y}-\mathbf{z}$ correspond to (r_2,θ_2). We have that

$$\widehat{\mathbf{x}-\mathbf{z}} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} \quad \text{and} \quad \widehat{\mathbf{y}-\mathbf{z}} = \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix},$$

and then the expression for $\nabla\phi$ (2.1) implies that

$$\nabla\phi = \begin{bmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{bmatrix} \tag{2.3}$$

$$= \begin{bmatrix} \cos\theta_1(1-\cos(\theta_1-\theta_2)) - \sin\theta_1\sin(\theta_1-\theta_2) \\ \sin\theta_1(1-\cos(\theta_1-\theta_2)) + \cos\theta_1\sin(\theta_1-\theta_2) \end{bmatrix}. \tag{2.4}$$

The quantity of interest, $\nabla_\Gamma \phi$, is the first component of these last two expressions. [1]

[1] An alternative expression for $\nabla\phi$ is $[-2\sin((\theta_1+\theta_2)/2)\sin((\theta_1-\theta_2)/2),\ 2\sin((\theta_1-\theta_2)/2)\cos((\theta_1+\theta_2)/2)]^T$, and the asymptotics (2.5) and (2.6) below can also be obtained using this expression.

We now obtain expressions for the asymptotics of $\nabla_\Gamma \phi$ as ε and $\delta \to 0$ in the following two situations: (a) $\mathbf{x}$ and $\mathbf{y}$ tend to $\mathbf{z}$ (i.e. we are in one of Regimes 1–4), and (b) $\mathbf{x}$ and $\mathbf{y}$ do not tend to $\mathbf{z}$ (i.e. we are in Regime 5). When (b) holds, then θ_1 and θ_2 do not tend to zero but $|\theta_1 - \theta_2|$ does, and thus from (2.4) we obtain that

$$\nabla_\Gamma \phi = (\cos\theta_1)\frac{(\theta_1-\theta_2)^2}{2} - (\sin\theta_1)(\theta_1-\theta_2) + \mathscr{O}\big(|\theta_1-\theta_2|^3\big). \tag{2.5}$$

When (a) holds, there are three possibilities, (i) θ_1 and θ_2 are both in $(0,\pi/2)$, (ii) θ_1 and θ_2 are both in $(\pi/2,\pi)$, and (iii) one of θ_1 and θ_2 is in $(0,\pi/2)$ and the other is in $(\pi/2,\pi)$. In the rest of this proof we assume that, whenever we are in one of Regimes 1–4, (i) holds. By symmetry, the arguments when (i) holds apply when (ii) holds, and one can modify the arguments to obtain the result of this lemma (the bound (2.2)) when (iii) holds (indeed, under (iii), $\nabla_\Gamma \phi$ does not tend to zero in Regimes 1–4 since $\widehat{\mathbf{x}-\mathbf{z}}$ and $\widehat{\mathbf{y}-\mathbf{z}}$ point in opposite directions).

Therefore, in Regimes 1–4 we assume that θ_1 and θ_2 are both in $(0,\pi/2)$. The definitions of these regimes imply that both θ_1 and θ_2 tend to zero, and thus the expression (2.3) implies that

$$\nabla_\Gamma \phi = \frac{1}{2}(\theta_2^2-\theta_1^2) + \mathscr{O}(\theta_1^4) + \mathscr{O}(\theta_2^4). \tag{2.6}$$

We now seek to understand how θ_1 and θ_2 depend on ε and δ in Regimes 1–4. Let Γ in a neighbourhood of $\mathbf{z}$ be the graph of the function $f(\xi)$, with the point $\mathbf{z}$ corresponding to $\xi = 0$. The geometry of Γ implies that $f(0) = f'(0) = 0$ and $f''(\xi) > 0$ for ξ in a neighbourhood of zero (this last fact is because Γ has strictly positive curvature). Since Γ is C^2, f is C^2, and then Taylor's theorem implies that, given $\xi \in \mathbb{R}$, there exists an $\eta \in (0,\xi)$ such that

$$f(\xi) = \frac{1}{2} f''(\eta)\,\xi^2. \tag{2.7}$$

Since $f'' > 0$ in a neighbourhood of 0, there exist constants ρ, m, and M, all > 0, such that

$$m \le f''(\eta) \le M \quad \text{for all } |\eta| < \rho,$$

and thus, using (2.7),

$$\frac{1}{2} m\xi^2 \le f(\xi) \le \frac{1}{2} M\xi^2 \quad \text{for all } |\xi| < \rho. \tag{2.8}$$

The fact that $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ lie on Γ means that

$$r_j \sin\theta_j = f(r_j\cos\theta_j), \quad j = 1,2,$$

and therefore, by (2.8), we have that

$$\frac{1}{2} m(r_j\cos\theta_j)^2 \le r_j\sin\theta_j \le \frac{1}{2} M(r_j\cos\theta_j)^2 \tag{2.9}$$

for all r_j and θ_j sufficiently small.

When $r_j \to 0$ and we think of θ_j as a function of r_j, (2.9) implies that

$$r_j\sin\theta_j \sim (r_j\cos\theta_j)^2 \quad \text{as } r_j \to 0. \tag{2.10}$$

When $r_j \to 0$, θ_j must tend to either 0 or π. Since $\theta_j \in (0,\pi/2)$ we have $\theta_j \to 0$ and then $\sin\theta_j \sim \theta_j$, $\cos\theta_j \sim 1$. Using these asymptotics in (2.10) we have that

$$\theta_j \sim r_j \text{ as } r_j \to 0. \tag{2.11}$$

Since we know how r_j depends on ε and δ in Regimes 1–4, (2.11) tells us how θ_j depends on ε and δ.

We now consider each of the 5 regimes separately, and prove that the bound (2.2) holds in each of them.

Regime 1. By definition, in this regime $r_1 \sim \delta$ and $r_2 \sim \varepsilon$, and thus both r_1 and $r_2 \to 0$. Using (2.11) we have that $\theta_1 \sim \delta$ and $\theta_2 \sim \varepsilon$. Using (2.6) we then have that $\nabla_\Gamma \phi \sim \varepsilon^2$.

Regime 2. In this regime $r_1 \sim |\mathbf{z}-\mathbf{x}| \to 0$ and $r_2 \sim \varepsilon$. Since both r_1 and $r_2 \to 0$, we can use (2.11) and obtain that $\theta_1 \sim |\mathbf{z}-\mathbf{x}|$ and $\theta_2 \sim \varepsilon$. Using the asymptotics (2.6) and the fact that $|\mathbf{z}-\mathbf{x}| \ll \varepsilon$ we have that $\nabla_\Gamma \phi \sim \varepsilon^2$.

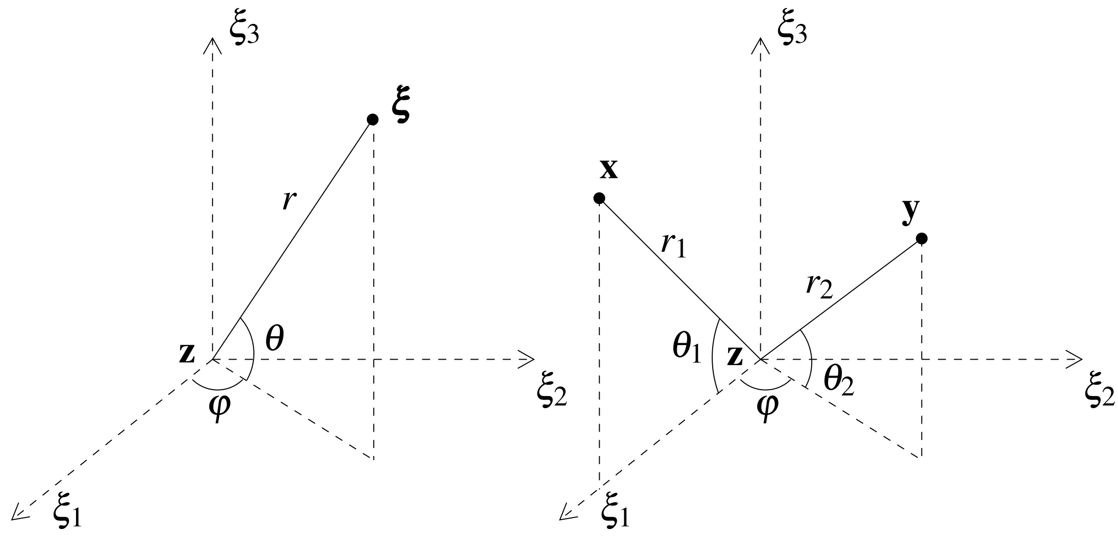

Fig. 2.2 The spherical polar coordinate system in 3–d in a neighbourhood of **z**

Regime 3. We have that $r_1 = a\varepsilon + o(1)$ and $r_2 = b\varepsilon + o(1)$ for some constants a and b with $a \neq b$ (a cannot be equal to b because $|\mathbf{x}-\mathbf{y}| = \varepsilon$). Since both r_1 and $r_2 \to 0$ we can use (2.11) to obtain that $\theta_1 \sim \varepsilon$, $\theta_2 \sim \varepsilon$, but $|\theta_2 - \theta_1| \gtrsim \varepsilon$. The asymptotics (2.6) then imply that $\nabla_\Gamma \phi \sim \varepsilon^2$.

Regime 4. We have that $\varepsilon \ll r_1 \ll 1$ and $r_1 \sim r_2$. By the triangle inequality $|r_1 - r_2| \leq \varepsilon$, but we now want to rule out the possibility that $|r_1 - r_2| \ll \varepsilon$. Since both r_1 and $r_2 \to 0$, $\mathbf{z}, \mathbf{x}$, and $\mathbf{y}$ are asymptotically collinear, and thus the fact that $|\mathbf{x}-\mathbf{y}| = \varepsilon$ implies that $|r_1 - r_2| \sim \varepsilon$ (see Figure 2.1). The asymptotics (2.11) then imply that $|\theta_1 - \theta_2| \sim \varepsilon$; hence $\theta_2^2 - \theta_1^2 \sim \varepsilon\theta_1 \sim \varepsilon|\mathbf{z}-\mathbf{x}|$. The asymptotics (2.6) and the fact that $|\mathbf{z}-\mathbf{x}| \gg \varepsilon$ then imply that $|\nabla_\Gamma \phi| \gtrsim \varepsilon^2$.

Regime 5. We have that $\theta_j \sim 1$ (since $r_j \sim 1$) and $\theta_1 - \theta_2 \to 0$. We therefore use the asymptotics (2.5), but we first need to determine how $\theta_1 - \theta_2$ depends on ε. Consider the triangle formed by $\mathbf{z}, \mathbf{x}$, and $\mathbf{y}$; by the cosine rule (or, equivalently, by expanding $|(\mathbf{z}-\mathbf{x}) - (\mathbf{z}-\mathbf{y})|^2$) we have that

$$\begin{aligned} \varepsilon^2 &= |\mathbf{z}-\mathbf{x}|^2 + |\mathbf{z}-\mathbf{y}|^2 - 2|\mathbf{z}-\mathbf{x}||\mathbf{z}-\mathbf{y}|\cos\theta_d, \\ &= r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_d, \end{aligned} \tag{2.12}$$

where $\theta_d := \theta_1 - \theta_2$. By the triangle inequality, $|r_1 - r_2| \leq \varepsilon$ (as we had in Regime 4), but now $|r_1 - r_2|$ could be $\ll \varepsilon$ (indeed, r_1 could even be equal to r_2). We therefore have that

$$\varepsilon^2 = 2r_1^2\big(1 - \cos\theta_d\big) + s, \tag{2.13}$$

where $|s| \lesssim \varepsilon r_1 \lesssim \varepsilon$ (since $r_1 \sim 1$). (Note that if $r_1 = r_2$ then $s = 0$.) If $0 \leq |s| \lesssim \varepsilon^2$ then (2.13) implies that $1 - \cos\theta_d \sim \varepsilon^2$ and thus $\theta_d \sim \varepsilon$. The asymptotics (2.5) then imply that $\nabla_\Gamma \phi \sim \varepsilon$ and thus $|\nabla_\Gamma \phi| \gg \varepsilon^2$. If $\varepsilon^2 \ll |s| \lesssim \varepsilon$ then (2.13) implies that $1 - \cos\theta_d \sim |s|$ and thus $\theta_d \sim |s|^{1/2}$. The asymptotics (2.5) then imply that $\nabla_\Gamma \phi \sim |s|^{1/2}$ and thus $|\nabla_\Gamma \phi| \gg \varepsilon \gg \varepsilon^2$.

In summary, in each of Regimes 1–5 we have shown that the bound (2.2) holds.

The case when Ω_- is 3–d. We introduce spherical polar coordinates at $\mathbf{z}$ as shown in Figure 2.2; thus, for a vector $\boldsymbol{\xi} \in \mathbb{R}^3$,

$$\xi_1 = r\cos\theta\cos\varphi, \quad \xi_2 = r\cos\theta\sin\varphi, \quad \xi_3 = r\sin\theta. \tag{2.14}$$

Note that the angle θ is different from its usual definition (the usual θ equals $\pi/2$ minus our θ); we make this change so that when $\varphi = 0$ (i.e. we restrict attention to the (ξ_1, ξ_3)–plane) we have the same coordinate system that we used in 2–d.

Let $\mathbf{x}-\mathbf{z}$ correspond to $(r_1,\theta_1,0)$ and $\mathbf{y}-\mathbf{z}$ correspond to (r_2,θ_2,φ); see Figure 2.2. Then

$$\widehat{\mathbf{x}-\mathbf{z}} = \begin{bmatrix}\cos\theta_1\\ 0\\ \sin\theta_1\end{bmatrix}, \qquad \widehat{\mathbf{y}-\mathbf{z}} = \begin{bmatrix}\cos\theta_2\cos\varphi\\ \cos\theta_2\sin\varphi\\ \sin\theta_2\end{bmatrix},$$

and (from (2.1))

$$\nabla\phi = \begin{bmatrix}\cos\theta_1-\cos\theta_2\cos\varphi\\ -\cos\theta_2\sin\varphi\\ \sin\theta_1-\sin\theta_2\end{bmatrix};$$

we therefore have that

$$\nabla_\Gamma\phi = \begin{bmatrix}\cos\theta_1-\cos\theta_2+\cos\theta_2(1-\cos\varphi)\\ -\cos\theta_2\sin\varphi\end{bmatrix}. \tag{2.15}$$

As we did in the 2–d case, for a point $\boldsymbol{\xi}=(r,\theta,\varphi)$ on the surface Γ, we now seek to understand how r, θ, and φ depend on each other as $\boldsymbol{\xi}\to\mathbf{z}$. Let Γ in a neighbourhood of $\mathbf{z}$ be the graph of the function $f(\boldsymbol{\xi})=f(\xi_1,\xi_2)$, with the point $\mathbf{z}$ corresponding to $(0,0)$, and the (ξ_1,ξ_2)-plane coinciding with the tangent plane to Γ at $\mathbf{z}$. (To maintain analogy with the 2–d case, we use $\boldsymbol{\xi}$ to denote a generic point in either $\mathbb{R}^2$ or $\mathbb{R}^3$, depending on the context.) The geometry of Γ implies that f is C^2, $f(0,0)=\partial_1 f(0,0)=\partial_2 f(0,0)=0$, and both eigenvalues of the quadratic form

$$\begin{bmatrix}\partial_1^2 f(0,0) & \partial_1\partial_2 f(0,0)\\ \partial_1\partial_2 f(0,0) & \partial_2^2 f(0,0)\end{bmatrix}$$

are >0 (these eigenvalues are the principal curvatures at $\mathbf{z}$).

Taylor's theorem implies that, given $\boldsymbol{\xi}\in\mathbb{R}^2$, there exists an $\boldsymbol{\eta}$ in the line segment $(\mathbf{0},\boldsymbol{\xi})$ such that

$$f(\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{\xi}^T\begin{bmatrix}\partial_1^2 f(\boldsymbol{\eta}) & \partial_1\partial_2 f(\boldsymbol{\eta})\\ \partial_1\partial_2 f(\boldsymbol{\eta}) & \partial_2^2 f(\boldsymbol{\eta})\end{bmatrix}\boldsymbol{\xi}; \tag{2.16}$$

see, e.g., [1, Theorem 12.14]. (Compare (2.16) to its 2–d analogue (2.7).) The fact that the quadratic form on the right-hand side of (2.16) is positive-definite when $\boldsymbol{\eta}=\mathbf{0}$ implies that there exist constants $\rho, a_\pm, b_\pm$, and $d_\pm$, with

$$a_\pm>0,\quad d_\pm>0,\quad\text{and}\quad 0\le 4(a_\pm d_\pm-b_\pm^2)\le(a_\pm+d_\pm)^2, \tag{2.17}$$

such that

$$\frac{1}{2}\boldsymbol{\xi}^T\begin{bmatrix}a_- & b_-\\ b_- & d_-\end{bmatrix}\boldsymbol{\xi}\le f(\boldsymbol{\xi})\le\frac{1}{2}\boldsymbol{\xi}^T\begin{bmatrix}a_+ & b_+\\ b_+ & d_+\end{bmatrix}\boldsymbol{\xi}\quad\text{for all } |\boldsymbol{\xi}|<\rho \tag{2.18}$$

(the conditions in (2.17) ensure that the two quadratic forms in (2.18) are positive-definite).

Now let $\boldsymbol{\xi}$ be a point on Γ (with polar coordinates (r,θ,φ)). Using (2.14) and the fact that $\xi_3=f(\xi_1,\xi_2)$, we have from (2.18) that

$$\frac{r^2(\cos\theta)^2}{2}\Big[a_-(\cos\varphi)^2+2b_-(\cos\varphi)(\sin\varphi)+d_-(\sin\varphi)^2\Big]$$
$$\le r\sin\theta\le\frac{r^2(\cos\theta)^2}{2}\Big[a_+(\cos\varphi)^2+2b_+(\cos\varphi)(\sin\varphi)+d_+(\sin\varphi)^2\Big] \tag{2.19}$$

(compare these inequalities with those in the 2–d case given by (2.9)). The conditions in (2.17) mean that the quantities in square brackets in (2.19) are >0, and therefore $\theta\sim r$ as $r\to 0$.

We now consider the same 5 regimes that we considered in the 2–d case (recalling that we are still in Case (i), i.e. $|\mathbf{x}-\mathbf{y}|=\varepsilon$).

Regimes 1–4. First consider Regime 1. Here $r_1\sim\delta$ and $r_2\sim\varepsilon$. Both r_1 and $r_2\to 0$, and then, by (2.19), $\theta_j\sim r_j$. There are now three possibilities: $\varphi=0,\varphi\to 0$, and $\varphi\sim 1$. The worst-case scenario is when $\varphi=0$ (indeed, from (2.15) we see that if $\varphi\sim 1$ then both components of $\nabla_\Gamma\phi$ are ~ 1, and thus the bound $|\nabla_\Gamma\phi|\gtrsim\varepsilon^2$ certainly holds). Furthermore, when $\varphi=0$ the second component of $\nabla_\Gamma\phi$ equals zero, and the first component is equal to the single component of $\nabla_\Gamma\phi$ in the 2-d case. The results from Regime 1 in the 2–d case then imply that the first component of $\nabla_\Gamma\phi$ (i.e. the first component of (2.15)) $\sim\varepsilon^2$, so certainly $|\nabla_\Gamma\phi|\gtrsim\varepsilon^2$.

The bound in Regimes 2–4 follows in a similar way, again making use of the results from the relevant 2–d cases.

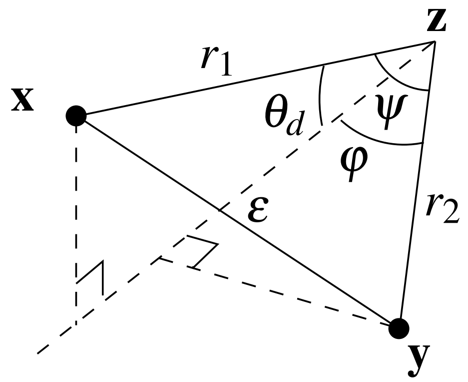

Fig. 2.3 The spherical polar coordinate system with $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$

Regime 5. Define ψ to be the (acute) angle between $\widehat{\mathbf{x}-\mathbf{z}}$ and $\widehat{\mathbf{y}-\mathbf{z}}$ (see Figure 2.3). The argument using the cosine rule used in Regime 5 of the 2–d case shows that $\varepsilon \lesssim \psi \lesssim \varepsilon^{1/2}$ as $\varepsilon \to 0$.

With $\theta_d := \theta_1 - \theta_2$ (as before), simple geometry implies that

$$\cos\theta_d \cos\varphi = \cos\psi. \tag{2.20}$$

(To prove (2.20), consider the tetrahedron formed by $\mathbf{z}, \mathbf{x}, \mathbf{y}$, and the projection of $\mathbf{x}$ onto the line corresponding to $\theta_d = 0$ and $\varphi = 0$, and calculate $|\mathbf{x}-\mathbf{y}|$ in terms of r_1, r_2, θ_d, ϕ, and ψ in two different ways.)

The equation (2.20) implies that if $\psi \to 0$, then both θ_d and φ tend to zero (since $\cos\alpha < 1$ for $0 < \alpha < \pi/2$), and then by Taylor expanding we find that

$$\psi^2 = \theta_d^2 + \varphi^2 + \mathscr{O}(\theta_d^2\varphi^2) + \mathscr{O}(\theta_d^4) + \mathscr{O}(\varphi^4) + \mathscr{O}(\psi^4).$$

Therefore, when $\psi \to 0$, at least one of θ_d and φ must $\sim \psi$. If $\varphi \nrightarrow 0$ then $\theta_d \sim \psi$ and the first component of $\nabla_\Gamma \phi$ tends to $\cos\theta_2(1-\cos\varphi)$, which ~ 1. If $\varphi \sim \psi$ then the modulus of the second component of $\nabla_\Gamma \phi \sim \psi$, which is $\gtrsim \varepsilon$. In either case, the modulus of one of the components of $\nabla_\Gamma \phi$ is $\gtrsim \varepsilon$, and thus $|\nabla_\Gamma \phi|$ is certainly $\gtrsim \varepsilon^2$.

Case (ii) We have now established the bound (2.2) in Cases (i) and (iii) (in both 2- and 3–d). To establish it in Case (ii), note that the only way ε entered the arguments for Case (i) above was as the norm of $\mathbf{x}-\mathbf{y}$. Thus, if $\varepsilon \ll |\mathbf{x}-\mathbf{y}| \ll 1$, the arguments in Case (i) show that $|\nabla_\Gamma \phi| \gtrsim |\mathbf{x}-\mathbf{y}|^2 \gtrsim \varepsilon^2$ and we are done.

3 Bounding the integral J_1

Given an arbitrary $\mathbf{x} \in \Gamma$ and $\varepsilon > 0$ we need to bound

$$\int_{\Gamma \cap B_\varepsilon(\mathbf{x})} \left| t_k(\mathbf{x}, \mathbf{y}) \right| \mathrm{d}s(\mathbf{y}),$$

where $t_k(\mathbf{x}, \mathbf{y})$ is given by (1.25).

The end result of this section is the bound (1.29) and we get this by proving that the bound (1.27) holds with $b(|\mathbf{x}-\mathbf{y}|, k, \varepsilon)$ given by (1.28).

Our goal, therefore, is to bound $t_k(\mathbf{x}, \mathbf{y})$, when $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, explicitly in $|\mathbf{x}-\mathbf{y}|$, k, and ε (although it turns out that the bound with not involve k).

Looking at the definition of $t_k(\mathbf{x}, \mathbf{y})$, (1.25), we see that if $\mathbf{x} = \mathbf{y}$ then the kernel of this integral is singular when $\mathbf{z} = \mathbf{x}$ (and thus $|t_k(\mathbf{x}, \mathbf{x})|$ is infinite), and even when $\mathbf{x} \neq \mathbf{y}$ the kernel is weakly singular when $\mathbf{z} = \mathbf{x}$ and when $\mathbf{z} = \mathbf{y}$. We begin by splitting the integral into two parts: one localised near the singularity at $\mathbf{z} = \mathbf{x}$, the other where the singularity is cut away. To make this split we introduce a cut-off function $\chi_{\delta,\mathbf{x}}$ such that

$$\chi_{\delta,\mathbf{x}}(\mathbf{z}) = 1 \qquad \text{for } \mathbf{z} \in B_\delta(\mathbf{x}), \tag{3.1a}$$

$$0 \le \chi_{\delta,\mathbf{x}}(\mathbf{z}) \le 1 \text{ and } \left|\partial_{\mathbf{z}}^m \left(\chi_{\delta,\mathbf{x}}(\mathbf{z})\right)\right| \lesssim \delta^{-|m|} \qquad \text{for } \mathbf{z} \in B_{2\delta}(\mathbf{x}) \setminus B_\delta(\mathbf{x}), \text{ and} \tag{3.1b}$$

$$\chi_{\delta,\mathbf{x}}(\mathbf{z}) = 0 \qquad \text{for } \mathbf{z} \notin B_{2\delta}(\mathbf{x}) \tag{3.1c}$$

(such a χ exists by, e.g., [19, Theorem 3.6]).

At the moment we do not specify how δ is related to ε, but we see later that the best bound is obtained when δ is independent of ε. In what follows we assume that ε and δ are both sufficiently small so that $\Gamma \cap B_\varepsilon(\mathbf{x})$ and $\Gamma \cap B_\delta(\mathbf{x})$ can both be expressed as the graphs of functions.

Let I_1 and I_2 be defined by

$$I_1(\mathbf{x},\mathbf{y}) := \int_\Gamma \Big(1-\chi_{\delta,\mathbf{x}}(\mathbf{z})\Big)\frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}),$$

and

$$I_2(\mathbf{x},\mathbf{y}) := \int_\Gamma \chi_{\delta,\mathbf{x}}(\mathbf{z})\frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}),$$

so that

$$t_k(\mathbf{x},\mathbf{y}) = \frac{1}{16\pi^2}\big(I_1(\mathbf{x},\mathbf{y})+I_2(\mathbf{x},\mathbf{y})\big).$$

Observe that the integrand of I_1 is zero when $\mathbf{z}\in B_\delta(\mathbf{x})$ and the integrand of I_2 is zero when $\mathbf{z}\notin B_{2\delta}(\mathbf{x})$. We now bound I_1 and I_2 separately (and the majority of the work occurs in bounding I_2).

3.1 Bounding I_1

We have that

$$\big|I_1(\mathbf{x},\mathbf{y})\big| \leq \int_{\Gamma\setminus B_\delta(\mathbf{x})}\frac{\mathrm{d}s(\mathbf{z})}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}.$$

The inequality $|\mathbf{z}-\mathbf{x}|\geq\delta$ for all $\mathbf{z}\in\Gamma\setminus B_\delta(\mathbf{x})$ implies that

$$\big|I_1(\mathbf{x},\mathbf{y})\big| \leq \frac{1}{\delta}\int_{\Gamma\setminus B_\delta(\mathbf{x})}\frac{\mathrm{d}s(\mathbf{z})}{|\mathbf{z}-\mathbf{y}|}\leq \frac{1}{\delta}\int_\Gamma\frac{\mathrm{d}s(\mathbf{z})}{|\mathbf{z}-\mathbf{y}|}.$$

Since $\int_\Gamma |\mathbf{z}-\mathbf{y}|^{-1}\mathrm{d}s(\mathbf{z})<\infty$, we have that

$$\big|I_1(\mathbf{x},\mathbf{y})\big| \lesssim \frac{1}{\delta}. \tag{3.2}$$

3.2 Bounding I_2

We have

$$\big|I_2(\mathbf{x},\mathbf{y})\big| \leq \int_{B_{2\delta}(\mathbf{x})}\frac{\mathrm{d}s(\mathbf{z})}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}.$$

Introduce polar coordinates in the tangent plane to Γ at $\mathbf{x}$. Let (r,θ) be the projection of $\mathbf{z}-\mathbf{x}$ into the tangent plane, and let $(\rho,0)$ be the projection of $\mathbf{y}$; see Figure 3.1. Since Γ is C^2 we have that, for δ and ε sufficiently small,

$$|\mathbf{x}-\mathbf{y}|\lesssim\rho\leq|\mathbf{x}-\mathbf{y}| \quad\text{and}\quad |\mathbf{z}-\mathbf{x}|\lesssim r\leq|\mathbf{z}-\mathbf{x}|. \tag{3.3}$$

If $\mathbf{y}\notin B_{2\delta}(\mathbf{x})$ then we obtain $|I_2|\lesssim 1/\delta$ in exactly the same way that we obtained the bound (3.2) on I_1. Therefore, in the rest of this section we assume that $2\delta>\varepsilon$ so that $B_\varepsilon(\mathbf{x})\subset B_{2\delta}(\mathbf{x})$ and thus $\mathbf{y}\in B_{2\delta}(\mathbf{x})$.

We now estimate $|I_2|$ in terms of ρ, and then use (3.3) to get an estimate in terms of $|\mathbf{x}-\mathbf{y}|$. For δ sufficiently small, we have that

$$|\mathbf{z}-\mathbf{y}|^2\sim r^2+\rho^2-2r\rho\cos\theta,$$

see, e.g., [8, Equation 2.5], and therefore

$$\big|I_2(\mathbf{x},\mathbf{y})\big| \lesssim \int_{\theta=0}^{2\pi}\int_{r=0}^{2\delta}\frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{r\sqrt{r^2+\rho^2-2r\rho\cos\theta}}. \tag{3.4}$$

We know that $|t_k(\mathbf{x},\mathbf{x})|$ is infinite, and this can be seen from (3.4), since when $r=\rho$ the integral in θ is singular. We perform the θ–integration in (3.4) first and find explicitly the singular behaviour when $r=\rho$,

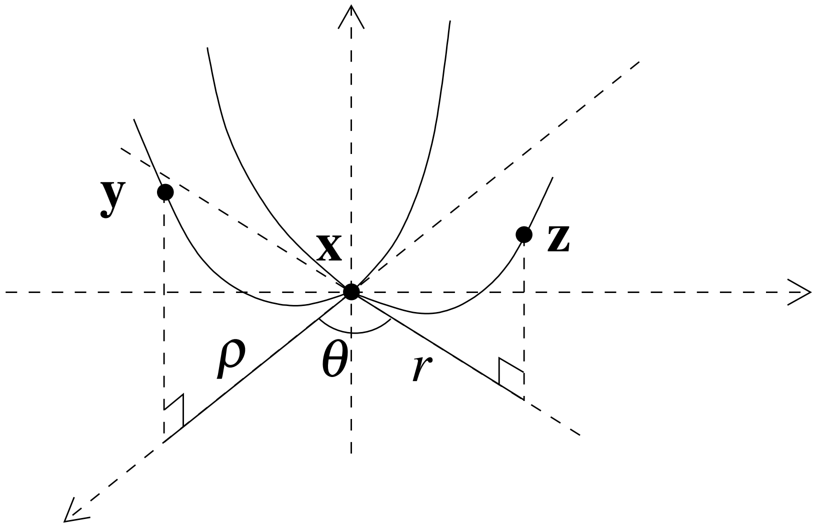

Fig. 3.1 The polar coordinate system used in the tangent plane to Γ at $\mathbf{x}$

but we first split the r–integral up to isolate this singularity. Indeed, given $\rho \in (0, 2\pi)$, we introduce an r_0 such that $0 < r_0 < \rho$ and $r_0 < 2\delta - \rho$, and split the integral as follows

$$\left|I_2(\mathbf{x},\mathbf{y})\right| \lesssim \left(\int_0^{\rho-r_0} + \int_{\rho-r_0}^{\rho+r_0} + \int_{\rho+r_0}^{2\delta}\right) \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \, \frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}}$$

(we fix r_0 as a specific function of ρ later). Call the three integrals on the right-hand side of the last inequality I_{21}, I_{22}, and I_{23} respectively. (To keep the expressions concise, we suppress the dependence of I_{21}, I_{22}, and I_{23} on $\mathbf{x}$ and $\mathbf{y}$ in the rest of the argument.)

Bounding the integrals I_{21} and I_{23}. We have that

$$I_{21} = \int_0^{\rho-r_0} \mathrm{d}r \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}}.$$

If $r < \rho$ then $\sqrt{r^2+\rho^2-2r\rho\cos\theta} \geq (\rho - r)$, so

$$I_{21} \leq 2\pi \int_0^{\rho-r_0} \frac{\mathrm{d}r}{\rho - r} = 2\pi \log\left(\frac{\rho}{r_0}\right). \tag{3.5}$$

In an almost identical way,

$$I_{23} \leq 2\pi \log\left(\frac{2\delta-\rho}{r_0}\right). \tag{3.6}$$

Bounding the integral I_{22}. We have that

$$I_{22} = \int_{\rho-r_0}^{\rho+r_0} F\left(\sqrt{\frac{2r\rho}{r^2+\rho^2}}\right) \frac{\mathrm{d}r}{\sqrt{r^2+\rho^2}}, \tag{3.7}$$

where

$$F(b) := \int_0^{2\pi} \frac{1}{\sqrt{1-b^2\cos\theta}} \, \mathrm{d}\theta.$$

Observe that $\sqrt{2r\rho/(r^2+\rho^2)}$ is always ≤ 1 and equal to 1 when $r = \rho$. Therefore we are interesting in the asymptotics of $F(b)$ as $b \nearrow 1$.

In Appendix A we prove that

$$\left|F(b) - \sqrt{2}\log\left(\frac{1}{1-b}\right)\right| \lesssim 1 \quad \text{as } b \nearrow 1,$$

and thus there exists a b_0 such that

$$F(b) \lesssim \sqrt{2}\log\left(\frac{1}{1-b}\right) \qquad \text{for all } b \text{ satisfying } 1-b_0 \le b < 1.$$

Now

$$\sqrt{\frac{2r\rho}{r^2+\rho^2}} = \sqrt{1-\frac{(r-\rho)^2}{r^2+\rho^2}} \le 1 - \frac{1}{2}\frac{(r-\rho)^2}{r^2+\rho^2},$$

so

$$\left(1-\sqrt{\frac{2r\rho}{r^2+\rho^2}}\right)^{-1} \le \frac{2(r^2+\rho^2)}{(r-\rho)^2},$$

and

$$\log\left(1-\sqrt{\frac{2r\rho}{r^2+\rho^2}}\right)^{-1} \le \log\left(\frac{2(r^2+\rho^2)}{(r-\rho)^2}\right).$$

Therefore, there exists a $c>0$ such that if $r_0 \le c$ (i.e. r_0 is sufficiently small) then

$$F\left(\sqrt{\frac{2r\rho}{r^2+\rho^2}}\right) \lesssim \log\left(\frac{1}{|r-\rho|}\right) \qquad \text{for all } |r-\rho| \le r_0. \tag{3.8}$$

Note that since $r_0 < \rho \le \varepsilon$, the condition "$r_0$ is sufficiently small" required to use (3.8) can be subsumed into the condition that ε is sufficiently small (with this latter condition applicable to all the bounds in this section).

Using the inequality (3.8) in the expression (3.7), we have that

$$\begin{aligned} I_{22} &\lesssim \int_{\rho-r_0}^{\rho+r_0} \log\left(\frac{1}{|r-\rho|}\right)\frac{\mathrm{d}r}{\sqrt{r^2+\rho^2}}, \sim \int_{\rho}^{\rho+r_0} \log\left(\frac{1}{r-\rho}\right)\frac{\mathrm{d}r}{\sqrt{r^2+\rho^2}}, \\ &\sim \int_0^{r_0} \log\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{\sqrt{2\rho^2+s^2+2\rho s}}, \end{aligned}$$

where we have used the change of variable $r=\rho+s$ in the last step. Since $\sqrt{2\rho^2+s^2+2\rho s} \ge \rho\sqrt{2}$ we have

$$I_{22} \lesssim \frac{r_0}{\rho}\left(1+\log\left(\frac{1}{r_0}\right)\right). \tag{3.9}$$

Therefore, putting the bounds on I_{21}, I_{22}, and I_{23} ((3.5), (3.9), and (3.6) respectively) together we obtain

$$|I_2| \lesssim \log\left(\frac{\rho}{r_0}\right) + \frac{r_0}{\rho}\left(1+\log\left(\frac{1}{r_0}\right)\right) + \log\left(\frac{2\delta-\rho}{r_0}\right), \tag{3.10}$$

with $r_0 < \rho \le \varepsilon < 2\delta$.

We now gear r_0 to ρ in such a way that $r_0 \ll \rho$; we take $r_0 = \rho^2$. We assume that $2\delta \le 1$, and then $\log((2\delta-\rho)/\rho^2) \lesssim \log(1/\rho^2) \lesssim \log(1/\rho)$. Therefore

$$|I_2| \lesssim \log\left(\frac{1}{\rho}\right) + \rho\left(1+\log\left(\frac{1}{\rho}\right)\right) \lesssim \log\left(\frac{1}{\rho}\right).$$

Finally, using $|\mathbf{x}-\mathbf{y}| \lesssim \rho \le |\mathbf{x}-\mathbf{y}|$ (from (3.3)) we obtain the result that there exists a $\delta_0 > 0$ such that

$$|I_2(\mathbf{x},\mathbf{y})| \lesssim \log\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \tag{3.11}$$

for $|\mathbf{x}-\mathbf{y}| \le \varepsilon < 2\delta < 2\delta_0$.

3.3 Finding a bound on J_1

Combining the bounds on I_1 and I_2, (3.2) and (3.11) respectively, we find that there exists a $\delta_0 > 0$ such that

$$\left|t_k(\mathbf{x},\mathbf{y})\right| \lesssim \log\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) + \frac{1}{\delta}, \tag{3.12}$$

for $|\mathbf{x}-\mathbf{y}| \le \varepsilon < 2\delta < 2\delta_0$.

For ε sufficiently small we can estimate the integral in $\mathbf{y}$ of $|t_k(\mathbf{x},\mathbf{y})|$ over $\Gamma \cap B_\varepsilon(\mathbf{x})$ by the integral in the tangent plane and obtain

$$\int_{\Gamma\cap B_\varepsilon(\mathbf{x})} \left|t_k(\mathbf{x},\mathbf{y})\right| \mathrm{d}s(\mathbf{y}) \lesssim \varepsilon^2 \log\left(\frac{1}{\varepsilon}\right) + \frac{\varepsilon^2}{\delta}$$

(where we have used that $\int_0^\varepsilon \rho \log(1/\rho)\,\mathrm{d}\rho = \mathscr{O}(\varepsilon^2\log(1/\varepsilon))$, and $\int_0^\varepsilon \rho\,\mathrm{d}\rho = \mathscr{O}(\varepsilon^2)$). Choosing δ to be independent of ε (but still less than δ_0), we obtain the bound (1.29).

4 Bounding J_2

Recalling the discussion in §1.5 and the inequality (1.30), given an arbitrary $\mathbf{x} \in \Gamma$ we need to bound

$$\max_{\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})} \left|t_k(\mathbf{x},\mathbf{y})\right|$$

where $t_k(\mathbf{x},\mathbf{y})$ is given by (1.25).

The end result of this section is the bound

$$\max_{\mathbf{y}\in\Gamma\setminus B_\varepsilon(\mathbf{x})} \left|t_k(\mathbf{x},\mathbf{y})\right| \lesssim \varepsilon^{n-1} + \frac{1}{k\varepsilon^{2n+5}}, \tag{4.1}$$

for any $n > 1$, from which the bound (1.31) on J_2 follows after using (1.30).

When $\mathbf{y} \notin B_\varepsilon(\mathbf{x})$, the kernel of $t_k(\mathbf{x},\mathbf{y})$ is not strongly singular, but it is weakly singular when $\mathbf{z} = \mathbf{x}$ or $\mathbf{z} = \mathbf{y}$. We begin by cutting away the singularity at $\mathbf{z} = \mathbf{x}$. Let $\chi_{\delta,\mathbf{x}}(\mathbf{z})$ satisfy (3.1) for some δ that we fix later (note that the δ in this section has no connection with the δ in §3). Let I_3 and I_4 be defined by

$$I_3(\mathbf{x},\mathbf{y}) := \int_\Gamma \chi_{\delta,\mathbf{x}}(\mathbf{z}) \frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z})$$

and

$$I_4(\mathbf{x},\mathbf{y}) := \int_\Gamma \left(1-\chi_{\delta,\mathbf{x}}(\mathbf{z})\right) \frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}),$$

so that

$$t_k(\mathbf{x},\mathbf{y}) = \frac{1}{16\pi^2}\big(I_3(\mathbf{x},\mathbf{y}) + I_4(\mathbf{x},\mathbf{y})\big).$$

Observe that the integrand of I_3 is zero when $\mathbf{z} \notin B_{2\delta}(\mathbf{x})$ and the integrand of I_4 is zero when $\mathbf{z} \in B_\delta(\mathbf{x})$.

4.1 Bounding the integral I_3

We have that

$$|I_3(\mathbf{x},\mathbf{y})| \le \int_{B_{2\delta}(\mathbf{x})} \frac{1}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\mathrm{d}s(\mathbf{z}). \tag{4.2}$$

We choose δ so that $2\delta < \varepsilon$. This implies that $\mathbf{y} \notin B_{2\delta}(\mathbf{x})$ and we then have the bound $|\mathbf{z}-\mathbf{y}| > \varepsilon - 2\delta$. Using this bound in (4.2) we find that

$$|I_3(\mathbf{x},\mathbf{y})| \le \frac{1}{\varepsilon - 2\delta}\int_{B_{2\delta}(\mathbf{x})} \frac{1}{|\mathbf{z}-\mathbf{x}|}\mathrm{d}s(\mathbf{z}).$$

If δ is sufficiently small then this last integral can be estimated in the tangent plane, with $\int_{B_{2\delta}(\mathbf{x})}|\mathbf{z}-\mathbf{x}|^{-1}\,\mathrm{d}s(\mathbf{z}) \lesssim 2\delta$. Therefore

$$|I_3(\mathbf{x},\mathbf{y})| \lesssim \frac{2\delta}{\varepsilon-2\delta} = \left(\frac{2\delta}{\varepsilon}\right)\left(\frac{1}{1-2\delta/\varepsilon}\right). \tag{4.3}$$

Observe that we need $\delta \ll \varepsilon$ to make this last expression go to zero. Therefore, we assume that $\delta \ll \varepsilon$ from this point on, and the bound (4.3) then becomes

$$|I_3(\mathbf{x},\mathbf{y})| \lesssim \frac{\delta}{\varepsilon}. \tag{4.4}$$

4.2 Bounding the integral I_4

The integrand in I_4 is not weakly singular when $\mathbf{z}=\mathbf{x}$, but it is when $\mathbf{z}=\mathbf{y}$. Let $\chi_{\delta,\mathbf{y}}(\mathbf{z})$ satisfy the conditions (3.1) with $\mathbf{x}$ replaced by $\mathbf{y}$. (Note that we could choose the radius of the ball around $\mathbf{y}$ to be different from that around $\mathbf{x}$, i.e. introduce a $\chi_{\theta,\mathbf{x}}(\mathbf{z})$ for some $\theta>0$, but it turns out that this is not advantageous.)

Let

$$I_{41}(\mathbf{x},\mathbf{y}) := \int_\Gamma \chi_{\delta,\mathbf{y}}(\mathbf{z})\Big(1-\chi_{\delta,\mathbf{x}}(\mathbf{z})\Big)\frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}),$$

and

$$I_{42}(\mathbf{x},\mathbf{y}) := \int_\Gamma \Big(1-\chi_{\delta,\mathbf{y}}(\mathbf{z})\Big)\Big(1-\chi_{\delta,\mathbf{x}}(\mathbf{z})\Big)\frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{z}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|)}}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|}\,\mathrm{d}s(\mathbf{z}),$$

and thus $I_4 = I_{41}+I_{42}$. (Similar to before, we suppress the dependence of I_{41} and I_{42} on $\mathbf{x}$ and $\mathbf{y}$ in the rest of the argument to keep the expressions concise.)

Turning first to I_{41}, we find that (in an almost identical way to how we obtained the bound (4.4) on $|I_3|$)

$$|I_{41}| \lesssim \frac{2\delta}{\varepsilon-2\delta} = \left(\frac{2\delta}{\varepsilon}\right)\left(\frac{1}{1-2\delta/\varepsilon}\right) \lesssim \frac{\delta}{\varepsilon}, \tag{4.5}$$

where the last inequality follows since we are assuming that $\delta \ll \varepsilon$.

Turning to I_{42}, we see that the range of integration can be changed from Γ to $\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$, since $\chi_{\delta,\mathbf{x}}(\mathbf{z})=1$ when $\mathbf{z}\in B_\delta(\mathbf{x})$ and $\chi_{\delta,\mathbf{y}}(\mathbf{z})=1$ when $\mathbf{z}\in B_\delta(\mathbf{y})$. The integral I_{42} is an oscillatory integral with no weak singularities, and thus we can integrate by parts. The integrand of I_{42} is of the form $\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}f(\mathbf{z})$ with $\phi(\mathbf{z})$ given by (1.35) and $f(\mathbf{z})$ given by (1.36) (as in §1.5, we suppress the dependence of ϕ and f on $\mathbf{x}$ and $\mathbf{y}$). Now

$$\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}f(\mathbf{z}) = \frac{1}{\mathrm{i}k}\nabla_\Gamma\left(\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\right)\cdot\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z}),$$

and, by (1.41),

$$\nabla_\Gamma\cdot\left(\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\right) = \nabla_\Gamma\left(\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\right)\cdot\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z}) + \mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\nabla_\Gamma\cdot\left(\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\right).$$

Therefore, using the two previous equations and Lemma 1.1 we have that, for $S\subset\Gamma$,

$$\begin{aligned}\int_S \mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}f(\mathbf{z})\,\mathrm{d}s(\mathbf{z}) &= \frac{1}{\mathrm{i}k}\int_S\nabla_\Gamma\left(\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\right)\cdot\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\,\mathrm{d}s(\mathbf{z})\\ &= \frac{1}{\mathrm{i}k}\int_{\partial S}\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\cdot\big(\boldsymbol{\tau}(\mathbf{z})\times\mathbf{n}(\mathbf{z})\big)\,\mathrm{d}\sigma(\mathbf{z})\\ &\qquad -\frac{1}{\mathrm{i}k}\int_S\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}\nabla_\Gamma\cdot\left(\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\right)\mathrm{d}s(\mathbf{z}),\end{aligned}$$

where $\boldsymbol{\tau}(\mathbf{z})$ is the unit tangent vector to ∂S (with orientation as described above Lemma 1.1) and $\mathrm{d}\sigma(\mathbf{z})$ is the 1–d measure on the curve ∂S.

We apply this last formula with $S=\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$. Thus, $I_{42}=I_{421}+I_{422}$ where

$$I_{421} := \left(\int_{\Gamma\cap\partial B_\delta(\mathbf{x})}+\int_{\Gamma\cap\partial B_\delta(\mathbf{y})}\right)\frac{\mathrm{e}^{\mathrm{i}k\phi(\mathbf{z})}}{\mathrm{i}k}\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\cdot\big(\boldsymbol{\tau}(\mathbf{z})\times\mathbf{n}(\mathbf{z})\big)\,\mathrm{d}\sigma(\mathbf{z})$$

and

$$I_{422} := -\int_{\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))} \frac{e^{ik\phi(\mathbf{z})}}{ik}\nabla_\Gamma\cdot\left(\frac{\nabla_\Gamma\phi(\mathbf{z})}{|\nabla_\Gamma\phi(\mathbf{z})|^2}f(\mathbf{z})\right)\mathrm{d}s(\mathbf{z}),$$

To bound the integrals I_{421} and I_{422} we require bounds on f and $\nabla_\Gamma f$.

Bounds on f and $\nabla_\Gamma f$. We claim that

$$\frac{1}{|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|} \lesssim \frac{1}{\varepsilon\delta} \tag{4.6}$$

when $\mathbf{z}\in\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$ and $|\mathbf{x}-\mathbf{y}|\geq\varepsilon$. (Note that one can easily prove that the left-hand side of (4.6) is $\lesssim \delta^{-2}$ from the inequalities $|\mathbf{z}-\mathbf{x}|\geq\delta$ and $|\mathbf{z}-\mathbf{y}|\geq\delta$.)

To prove this claim, we first assume that $|\mathbf{z}-\mathbf{x}|\leq|\mathbf{z}-\mathbf{y}|$, i.e. $\mathbf{z}$ is always closer to $\mathbf{x}$ than to $\mathbf{y}$ (since the expression is symmetric in $\mathbf{x}$ and $\mathbf{y}$, this is without loss of generality). We can divide the set $\{\mathbf{z}:\mathbf{z}\in\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))\}$ into the following 3 regimes.

Regime 1. $\delta\leq|\mathbf{z}-\mathbf{x}|\ll\varepsilon$,
Regime 2. $|\mathbf{z}-\mathbf{x}|\sim\varepsilon$,
Regime 3. $\varepsilon\ll|\mathbf{z}-\mathbf{x}|\lesssim 1$.

In Regime 1, we have

$$|\mathbf{z}-\mathbf{y}|\geq|\mathbf{x}-\mathbf{y}|-|\mathbf{z}-\mathbf{x}|\geq\varepsilon-|\mathbf{z}-\mathbf{x}|\gtrsim\varepsilon,$$

and then the bound (4.6) follows from this last inequality and $|\mathbf{z}-\mathbf{x}|\geq\delta$.

In Regime 2, the triangle inequality and the assumption that $|\mathbf{z}-\mathbf{x}|\leq|\mathbf{z}-\mathbf{y}|$ imply that $|\mathbf{z}-\mathbf{y}|\sim\varepsilon$; thus $|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|\sim\varepsilon^2$ and the bound (4.6) certainly holds.

In Regime 3, we have that $|\mathbf{z}-\mathbf{y}|\sim|\mathbf{z}-\mathbf{x}|$, so $|\mathbf{z}-\mathbf{y}||\mathbf{z}-\mathbf{x}|\gg\varepsilon^2\gg\varepsilon\delta$; the bound (4.6) therefore holds in this regime too, and we have proved the claim.

The bound (4.6) implies that $|f(\mathbf{z})|\lesssim(\varepsilon\delta)^{-1}$ for all $\mathbf{z}\in\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y}))$ and $|\mathbf{x}-\mathbf{y}|\geq\varepsilon$. We now turn our attention to $\nabla_\Gamma f(\mathbf{z})$. Define $f_\mathbf{y}(\mathbf{z})$ by

$$f_\mathbf{y}(\mathbf{z}) := \frac{1-\chi_{\delta,\mathbf{y}}(\mathbf{z})}{|\mathbf{z}-\mathbf{y}|},$$

and define $f_\mathbf{x}(\mathbf{z})$ similarly. Then $f(\mathbf{z})=f_\mathbf{y}(\mathbf{z})f_\mathbf{x}(\mathbf{z})$ and

$$\nabla f(\mathbf{z}) = \Big(\nabla f_\mathbf{y}(\mathbf{z})\Big)f_\mathbf{x}(\mathbf{z}) + f_\mathbf{y}(\mathbf{z})\Big(\nabla f_\mathbf{x}(\mathbf{z})\Big).$$

Now

$$\Big(\nabla f_\mathbf{y}(\mathbf{z})\Big)f_\mathbf{x}(\mathbf{z}) = \left(\frac{\nabla(1-\chi_{\delta,\mathbf{y}}(\mathbf{z}))}{|\mathbf{z}-\mathbf{y}|} - \big(1-\chi_{\delta,\mathbf{y}}(\mathbf{z})\big)\frac{\mathbf{z}-\mathbf{y}}{|\mathbf{z}-\mathbf{y}|^3}\right)\frac{1-\chi_{\delta,\mathbf{x}}(\mathbf{z})}{|\mathbf{z}-\mathbf{x}|}$$

Using the bounds $|\nabla\chi_{\delta,\mathbf{y}}(\mathbf{z})|\lesssim\delta^{-1}$ (from (3.1b)), (4.6), and $|\mathbf{z}-\mathbf{y}|\geq\delta$, we then have that

$$\left|\Big(\nabla f_\mathbf{y}(\mathbf{z})\Big)f_\mathbf{x}(\mathbf{z})\right| \lesssim \frac{1}{\delta^2\varepsilon}.$$

An identical argument shows that $|f_\mathbf{y}(\mathbf{z})\nabla f_\mathbf{x}(\mathbf{z})|$ is also $\lesssim(\delta^2\varepsilon)^{-1}$. Therefore, since $|\nabla_\Gamma f|\leq|\nabla f|$,

$$\big|\nabla_\Gamma f(\mathbf{z})\big| \lesssim \frac{1}{\delta^2\varepsilon} \quad \text{when } \mathbf{z}\in\Gamma\setminus(B_\delta(\mathbf{x})\cup B_\delta(\mathbf{y})) \text{ and } |\mathbf{x}-\mathbf{y}|\geq\varepsilon. \tag{4.7}$$

Bounding I_{421}. Using the bounds (4.6) and $|\nabla_\Gamma\phi|\gtrsim\varepsilon^2$ (with the latter bound coming from Lemma 2.2), we can bound the modulus of the integrand of I_{421} by $(k\varepsilon^3\delta)^{-1}$. Since the length of $\Gamma\cap\partial B_\delta(\mathbf{x})$ is proportional to δ, the integral over $\Gamma\cap\partial B_\delta(\mathbf{x})$ in I_{421} is bounded by $(k\varepsilon^3)^{-1}$. The integral over $\Gamma\cap\partial B_\delta(\mathbf{y})$ is bounded in an identical way, resulting in the bound

$$|I_{421}| \lesssim \frac{1}{k\varepsilon^3}. \tag{4.8}$$

Bounding I_{422}. We have that

$$\nabla_\Gamma \cdot \left(\frac{\nabla_\Gamma \phi(\mathbf{z})}{|\nabla_\Gamma \phi(\mathbf{z})|^2} f(\mathbf{z}) \right) = \nabla_\Gamma f(\mathbf{z}) \cdot \frac{\nabla_\Gamma \phi(\mathbf{z})}{|\nabla_\Gamma \phi(\mathbf{z})|^2} + \frac{\Delta_\Gamma \phi(\mathbf{z})}{|\nabla_\Gamma \phi(\mathbf{z})|^2} f(\mathbf{z}) - 2\frac{\nabla_\Gamma \phi(\mathbf{z})}{|\nabla_\Gamma \phi(\mathbf{z})|^3} \cdot \nabla_\Gamma \Big(\big|\nabla_\Gamma \phi(\mathbf{z})\big| \Big) f(\mathbf{z}),$$

where $\Delta_\Gamma \phi := \nabla_\Gamma \cdot \nabla_\Gamma \phi$; see, e.g., [23, Equation 2.5.191]. Denote the integrals arising from the three terms on the right-hand side by I_{4221}, I_{4222}, and I_{4223} respectively.

Using the bounds (4.7) and $|\nabla_\Gamma \phi| \gtrsim \varepsilon^2$, we have that

$$|I_{4221}| \lesssim \frac{1}{k\varepsilon^3 \delta^2}. \tag{4.9}$$

To bound I_{4222} we need to bound $\Delta_\Gamma \phi(\mathbf{z})$. Since $\phi(\mathbf{z})$ is differentiable in a neighbourhood of $\mathbf{z} \in \Gamma \setminus (B_{\delta,\mathbf{x}}(\mathbf{z}) \cup B_{\delta,\mathbf{y}}(\mathbf{z}))$, $\Delta_\Gamma \phi(\mathbf{z})$ can be bounded by the first and second derivatives of ϕ in the domain, i.e.

$$\big|\Delta_\Gamma \phi(\mathbf{z})\big| \lesssim \sum_{|\alpha|\le 2} \big|\partial^\alpha \phi(\mathbf{z})\big| \quad \text{for } \mathbf{z} \in \Gamma, \tag{4.10}$$

where the omitted constant depends on the curvature (and thus relies on Γ being C^2); see, e.g., [23, Equation 2.5.212]. Therefore

$$\big|\Delta_\Gamma \phi(\mathbf{z})\big| \lesssim 1 + \frac{1}{|\mathbf{z}-\mathbf{y}|} + \frac{1}{|\mathbf{z}-\mathbf{x}|} \lesssim \frac{1}{\delta} \quad \text{when } \mathbf{z} \in \Gamma \setminus (B_\delta(\mathbf{x}) \cup B_\delta(\mathbf{y})) \text{ and } |\mathbf{x}-\mathbf{y}| \ge \varepsilon.$$

Using this last bound along with the bounds (4.6) and $|\nabla_\Gamma \phi| \gtrsim \varepsilon^2$, we have that

$$|I_{4222}| \lesssim \frac{1}{k\varepsilon^5 \delta^2}. \tag{4.11}$$

Finally, to bound I_{4223} we need to bound $\nabla_\Gamma(|\nabla_\Gamma \phi|)$. Similar to (4.10) we have that

$$\Big|\nabla_\Gamma \Big(\big|\nabla_\Gamma \phi(\mathbf{z})\big| \Big)\Big| \lesssim \sum_{|\alpha|\le 2} \big|\partial^\alpha \phi(\mathbf{z})\big| \quad \text{for } \mathbf{z} \in \Gamma,$$

and thus, in a similar manner to how we obtained the bound (4.11) on I_{4222}, we obtain that

$$|I_{4223}| \lesssim \frac{1}{k\varepsilon^5 \delta^2}. \tag{4.12}$$

Putting everything together. Combining the bounds on I_3, I_{41}, I_{421}, I_{4221}, I_{4222}, and I_{4223}, (4.4), (4.5) (4.8), (4.9), (4.11), and (4.12) respectively, we have that, when $\mathbf{x} \in \Gamma$ and $\mathbf{y} \in \Gamma \setminus B_\varepsilon(\mathbf{x})$,

$$\begin{aligned} \big|t_k(\mathbf{x},\mathbf{y})\big| &\lesssim \underbrace{\frac{\delta}{\varepsilon}}_{I_3} + \underbrace{\frac{\delta}{\varepsilon}}_{I_{41}} + \underbrace{\frac{1}{k\varepsilon^3}}_{I_{421}} + \underbrace{\frac{1}{k\varepsilon^3\delta^2}}_{I_{4221}} + \underbrace{\frac{1}{k\varepsilon^5\delta^2}}_{I_{4222}} + \underbrace{\frac{1}{k\varepsilon^5\delta^2}}_{I_{4223}} \\ &\lesssim \frac{\delta}{\varepsilon} + \frac{1}{k\varepsilon^5\delta^2}. \end{aligned} \tag{4.13}$$

Our only requirement on δ is that $\delta \ll \varepsilon$. We now let $\delta = \varepsilon^n$ for $n > 1$. Other choices of δ are available, e.g. we could choose $\delta = \varepsilon^n \log(1/\varepsilon)$ for $n > 1$, but these other choices do not result in a sharper bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ than the one resulting from the choice $\delta = \varepsilon^n$. With this choice of δ, (4.13) becomes (4.1), which is the end result of this section.

A The asymptotics of the integral $F(b)$ as $b \nearrow 1$

We use the following result to bound I_{22} in §3.2.

Lemma A.1 *If*

$$F(b) := \int_0^{2\pi} \frac{1}{\sqrt{1-b^2\cos\theta}}\,\mathrm{d}\theta$$

then

$$\left|F(b) - \sqrt{2}\log\left(\frac{1}{1-b}\right)\right| \lesssim 1 \quad \textit{as } b \nearrow 1. \tag{A.1}$$

Proof Define $I(\varepsilon)$ by

$$I(\varepsilon) := \int_0^{\pi} \frac{1}{\sqrt{1-(1-\varepsilon)^2\cos\theta}}\,\mathrm{d}\theta. \tag{A.2}$$

Then $I(\varepsilon) = F(1-\varepsilon)/2$, and proving that (A.1) holds is equivalent to proving that

$$\left|I(\varepsilon) - \frac{1}{\sqrt{2}}\log\left(\frac{1}{\varepsilon}\right)\right| \lesssim 1 \quad \text{as } \varepsilon \searrow 0. \tag{A.3}$$

The integrand of (A.2) behaves differently depending on whether θ is "large" compared to ε or θ is "small" compared to ε. Our plan is to use the "divide and conquer" technique of [17, §3.4], which involves breaking $I(\varepsilon)$ up into the sum of two integrals to separate these different behaviours.

Although this method can be applied to $I(\varepsilon)$ directly, it turns out that performing some elementary manipulations to the integral beforehand simplifies the calculations later on. Letting $x = \cos\theta$ in (A.2), we find that

$$I(\varepsilon) = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-(1-\varepsilon)^2 x}\sqrt{1-x^2}}.$$

We now let

$$I_1(\varepsilon) := \int_{-1}^{0} \frac{\mathrm{d}x}{\sqrt{1-(1-\varepsilon)^2 x}\sqrt{1-x^2}} \quad \text{and} \quad I_2(\varepsilon) := \int_{0}^{1} \frac{\mathrm{d}x}{\sqrt{1-(1-\varepsilon)^2 x}\sqrt{1-x^2}}$$

so that $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon)$. Since $I_1(0)$ is finite (as the singularity of the integrand when $\varepsilon = 0$ is at $x = 1$), it is straightforward to show that $|I_1(\varepsilon)| \lesssim 1$ as $\varepsilon \searrow 0$; we can therefore restrict attention to $I_2(\varepsilon)$.

The integrand of $I_2(0)$ is singular at $x = 1$; it is perhaps more convenient to have the singularity at zero, so we therefore let $t = 1-x$ and find that

$$I_2(\varepsilon) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-(1-\varepsilon)^2(1-t)}\sqrt{2-t}\sqrt{t}}.$$

Taylor's theorem implies that there exists a $C > 0$ such that

$$\left|\frac{1}{\sqrt{1-t/2}} - 1\right| \le Ct \quad \text{for all } t \in [0,1],$$

and thus

$$\left|I_2(\varepsilon) - \frac{1}{\sqrt{2}}J(\varepsilon)\right| \lesssim 1 \quad \text{as } \varepsilon \searrow 0,$$

where

$$J(\varepsilon) := \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-(1-\varepsilon)^2(1-t)}\sqrt{t}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2 t}\sqrt{t}}. \tag{A.4}$$

Therefore, to prove (A.3), it is sufficient to prove that

$$\left|J(\varepsilon) - \log\left(\frac{1}{\varepsilon}\right)\right| \lesssim 1 \quad \text{as } \varepsilon \searrow 0. \tag{A.5}$$

The second expression for $J(\varepsilon)$ in (A.4) shows that the behaviour of the integrand depends on the relative magnitudes of t and ε. Following the "divide and conquer" method discussed above, we introduce $\delta > 0$ and define

$$J_1(\varepsilon) := \int_0^{\delta} \frac{\mathrm{d}t}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2 t}\sqrt{t}} \quad \text{and} \quad J_2(\varepsilon) := \int_{\delta}^{1} \frac{\mathrm{d}t}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2 t}\sqrt{t}}$$

(so that $J(\varepsilon) = J_1(\varepsilon) + J_2(\varepsilon)$). The rest of the proof consists of showing that, if $\delta \gg \varepsilon$,

$$\left|J_1(\varepsilon) - \log\left(\frac{\delta}{\varepsilon}\right)\right| \lesssim 1 \quad \text{as } \varepsilon \searrow 0, \quad \text{and} \tag{A.6}$$

$$\left|J_2(\varepsilon) - \log\left(\frac{1}{\delta}\right)\right| \lesssim 1 \quad \text{as } \varepsilon \searrow 0, \tag{A.7}$$

and then (A.5) follows from combining (A.6) and (A.7).

Proof of (A.6). The integral $J_1(\varepsilon)$ can be computed exactly. Indeed, using the change of variable $t = s^2$, we find that

$$J_1(\varepsilon) = 2\int_0^{\sqrt{\delta}} \frac{\mathrm{d}s}{\sqrt{(2\varepsilon-\varepsilon^2)+(1-\varepsilon)^2 s^2}}.$$

The substitution $s = (a/b)\sinh x$ can be used to show that

$$\int_0^c \frac{\mathrm{d}s}{\sqrt{a^2+b^2s^2}} = \frac{1}{b}\log\left(\frac{bc}{a}+\sqrt{\left(\frac{bc}{a}\right)^2+1}\right) = \frac{1}{b}\left[\log\left(\frac{bc}{a}\right)+\log\left(1+\sqrt{1+\left(\frac{a}{bc}\right)^2}\right)\right].$$

Using this last expression with $a = \sqrt{2\varepsilon-\varepsilon^2}$, $b = 1-\varepsilon$, and $c = \sqrt{\delta}$, yields

$$J_1(\varepsilon) = \frac{2}{1-\varepsilon}\left[\log\left(\frac{(1-\varepsilon)\sqrt{\delta}}{\sqrt{2\varepsilon-\varepsilon^2}}\right)+\log\left(1+\sqrt{1+\frac{2\varepsilon-\varepsilon^2}{(1-\varepsilon)^2\delta}}\right)\right].$$

Assuming that $\delta \gg \varepsilon$, we then obtain the asymptotics (A.6).

Proof of (A.7). Since $t \geq \delta \gg \varepsilon$, we expect the integrand of $J_2(\varepsilon)$ to behave like $1/t$. We therefore prove that

$$\left|J_2(\varepsilon) - \int_\delta^1 \frac{1}{t}\mathrm{d}t\right| \lesssim 1,$$

and then (A.7) follows.

Now

$$J_2(\varepsilon) - \int_\delta^1 \frac{1}{t}\mathrm{d}t = \int_\delta^1 \frac{1}{\sqrt{t}}\left[\frac{1}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2t}} - \frac{1}{\sqrt{t}}\right]\mathrm{d}t, \tag{A.8}$$

and thus we need to bound the term in square brackets in (A.8).

Combining the fact that

$$\frac{1}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2t}} - \frac{1}{\sqrt{t}} = \frac{-(2\varepsilon-\varepsilon^2)(1-t)}{(\sqrt{t}+\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2t})\sqrt{t}\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2t},}$$

with the inequality

$$\frac{1}{\sqrt{2\varepsilon-\varepsilon^2+(1-\varepsilon)^2t}} \leq \frac{2}{\sqrt{t}} \quad \text{when } \varepsilon \leq \frac{1}{2},$$

we find that

$$\left|J_2(\varepsilon) - \int_\delta^1 \frac{1}{t}\mathrm{d}t\right| \lesssim (2\varepsilon-\varepsilon^2)\int_\delta^1 \frac{1-t}{t^2}\mathrm{d}t \lesssim \frac{\varepsilon}{\delta} \lesssim 1,$$

and we are done.

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