Recap

$(-k^{-2}\Delta - 1)u = f$ in $\mathbb{R}^d$ + r.c.

have $\|u\|_{H^2_k(B_R)} \lesssim C\, kR\, \|f\|_{L^2(B_R)}$

Thm (Melenk + Sauter 2010)

components of u with freq $\geq \lambda k$, $\lambda > 1$

$\lesssim \lambda k$

$$u|_{B_R} = u_{H^2} + u_{\mathcal{A}}$$

where $\|u_{H^2}\|_{H^2_k(B_R)} \lesssim C_1 \|f\|_{L^2(B_R)}$ $\forall k \geq k_0$

$$\|(k^{-1}\partial)^\alpha u_{\mathcal{A}}\|_{L^2(B_R)} \lesssim C_2\, kR\, (C_3)^{|\alpha|} \|f\|_{L^2(B_R)} \quad \forall k \geq k_0,\ \forall \alpha$$

Fourier transform

$$\mathcal{F}_k \phi(\xi) := \int_{\mathbb{R}^d} e^{-ikx\cdot\xi} \phi(x)\, dx$$

$$\mathcal{F}_h^{-1} \psi(x) := \left(\frac{k}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ikx\cdot\xi} \psi(\xi)\, d\xi$$

$$\mathcal{F}_k\Big(\big(\underbrace{-ik^{-1}\partial}_{k^{-1}D}\big)^\alpha \phi\Big)(\xi) = \xi^\alpha \mathcal{F}_h \phi(\xi)$$

$-i\partial = D$

$$\|\phi\|_{L^2(\mathbb{R}^d)} = \left(\frac{k}{2\pi}\right)^d \|\mathcal{F}_h \phi\|_{L^2(\mathbb{R}^d)}$$

$$\|\phi\|^2_{H^s_h(\mathbb{R}^d)} = \left(\frac{k}{2\pi}\right)^d \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\mathcal{F}_h \phi(\xi)|^2\, d\xi$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$

Fourier multipliers

$$\boxed{\left(a(h^{-1}D)v\right)(x) := \mathcal{F}_h^{-1}\left(a(\xi)\left(\mathcal{F}_h v\right)(\xi)\right)(x)}$$

$\xi \mapsto x$

motivation: $-h^{-2}\Delta - 1 = p(h^{-1}D)$ when $p(\xi) = |\xi|^2 - 1$ e.g. $p \in (FS)^2$

Def^n (non standard) a is a Fourier symbol of order $m \in \mathbb{R}$ if $\exists\, C > 0$ st

$$|a(\xi)| \leq C \langle \xi \rangle^m \quad \forall \xi$$

, write $a \in (FS)^m$

e.g. 2 $a(\xi) = 1$ then $a \in (FS)^0$ $\quad a(h^{-1}D)v(x) = v(x)$

e.g. 3 supp a compact then $a \in (FS)^{-N}$ $\forall N > 1$, $1 - a \in (FS)^0$

Lemma $a \in (FS)^{m_a}$, $b \in (FS)^{m_b}$ $\Rightarrow$ $ab \in (FS)^{m_a + m_b}$

$$a(h^{-1}D)b(h^{-1}D) = (ab)(h^{-1}D) = b(h^{-1}D)a(h^{-1}D)$$

$$\|a(h^{-1}D)v\|^2_{H_h^{s-m_a}} = \int \langle \xi \rangle^{2s-2m_a} |a(\xi)\mathcal{F}_h v(\xi)|^2$$

$$\leq C \int \langle \xi \rangle^{2s} |\mathcal{F}_h v(\xi)|^2 d\xi$$

$\rightarrow$ $a(h^{-1}D) : H_h^s \to H_h^{s-m_a}$ with $\|a(h^{-1}D)\|_{H_h^s \to H_h^{s-m_a}} \leq C$

pf of thm

let $\chi_\lambda(\xi) := \mathbb{1}_{|\xi| \leq \lambda}(\xi)$

$$\Pi_L := \chi_\lambda(h^{-1}D) \quad \text{i.e.} \quad \Pi_L v = \mathcal{F}_h^{-1}\left(\chi_\lambda(\cdot)(\mathcal{F}_h v)(\cdot)\right)$$

low freq. cutoff

returns freq. $\leq \lambda$

$$\Pi_H := I - \Pi_L = (1 - \chi_\lambda)(h^{-1}D)$$

returns freq. $\geq \lambda$

$\varphi \in C^\infty_{comp}(\mathbb{R}^d; [0,1])$ $\quad \varphi \equiv 1$ on B_R, $\operatorname{supp} \varphi \subset B_{2R}$

$$u_{\mathcal{L}} := \Pi_L(\varphi u)\big|_{B_R}, \quad u_{H^2} := \Pi_H(\varphi u)\big|_{B_R}$$

on B_R $\quad u_{\mathcal{L}} + u_{H^2} = \varphi u = u$

pf of bound on u_A

function with compactly supported Fourier transform is analytic

$$\|(k^{-1}\partial)^\alpha u_A\|_{L^2(B_R)} = \|(k^{-1}\partial)^\alpha \Pi_L(\varphi u)\|_{L^2(B_R)} \leq \|(k^{-1}\partial)^\alpha \Pi_L(\varphi u)\|_{L^2(\mathbb{R}^d)}$$

$$= \left(\frac{k}{2\pi}\right)^d \|\underbrace{\xi^\alpha \chi_\lambda(\xi)}_{\leq \lambda^{|\alpha|}} \mathcal{F}_k(\varphi u)(\xi)\|_{L^2(\mathbb{R}^d)}$$

$$\leq \left(\frac{k}{2\pi}\right)^d \lambda^{|\alpha|} \|\mathcal{F}_k(\varphi u)\|_{L^2(\mathbb{R}^d)}$$

$$= \lambda^{|\alpha|} \|\varphi u\|_{L^2(\mathbb{R}^d)}$$

$$\leq \lambda^{|\alpha|} \|u\|_{L^2(B_{2R})} \leq \lambda^\alpha C kR \|f\|_{L^2}$$

a priori bound

$$\boxed{\|(k^{-1}\partial)^\alpha u_A\|_{L^2(B_R)} \leq C_2 kR (C_3)^{|\alpha|} \|f\|_{L^2(B_R)}}$$

$C_3 = \lambda$

$C_2 = C$

pf of bound on u_{H^2}

$$\boxed{\|u_{H^2}\|_{H^2_k(B_R)} \le C_1 \|f\|_{L^2(B_R)}}$$

$$\|u_{H^2}\|_{H^2_k(B_R)} = \|\Pi_H(\varphi u)\|_{H^2_k(B_R)} \le \|\Pi_H(\varphi u)\|_{H^2_k(\mathbb{R}^d)}$$

if $\lambda \ge \lambda_0 > 1$ then $\exists C > 0$ s.t.

$$\underbrace{\left|\,|\xi|^2 - 1\,\right|}_{p(\xi)} \ge C \langle \xi \rangle^2 \text{ for } |\xi| \ge \lambda$$

$(1 + |\xi|^2)$

$$= \|(1-\chi_\mu)(k^{-1}D)\,\varphi u\|_{H^2_k(\mathbb{R}^d)}$$

$$= \left\|\frac{(1-\chi_\mu)(k^{-1}D)}{p(k^{-1}D)}\, p(k^{-1}D)\,\varphi u\right\|_{H^2_k(\mathbb{R}^d)}$$

restricts to $|\xi| \ge \lambda$

$$\underbrace{\frac{(1-\chi_\mu)(\xi)}{p(\xi)}}_{} \in (FS)^{-2}$$

$(k^{-2}\Delta - 1)$

$$\le C \|p(k^{-1}D)\varphi u\|_{L^2(\mathbb{R}^d)}$$

$$= C\|\varphi f + [-k^{-2}\Delta - 1, \varphi]u\|_{L^2(\mathbb{R}^d)} \lesssim C\|f\|_{L^2(\mathbb{R}^d)} + \frac{1}{kR}\|u\|_{H^1_k(B_{2R})}$$

$$\le \tilde{C}\|f\|_{L^2(\mathbb{R}^d)}$$

$(k^{-2}\Delta - 1)\varphi u = \varphi(-k^{-2}\Delta - 1)u + [-k^{-2}\Delta - 1, \varphi]u$

Thm suppose $a \in (FS)^{m_a}$, $b \in (FS)^{m_b}$ and $\exists\, c>0$

s.t. $|b(\xi)| \geq c \langle \xi \rangle^{m_b}$ for $\xi \in \operatorname{supp} a$

then $a(k^{-1}D) = \underbrace{q(k^{-1}D)}\, b(k^{-1}D)$

$\in (FS)^{m_a - m_b}$ s.t. $q(\xi) = \dfrac{a(\xi)}{b(\xi)}$

means q makes sense

$a(\xi) = (1-\chi_\Lambda)(\xi)$

$b(\xi) = p(\xi) = |\xi|^2 - 1$

$\geq c\langle \xi \rangle^2$

on $|\xi| \geq \Lambda > 1$

$-k^{-2}\Delta - 1$ is semiclassically elliptic for $|\xi| > 1$

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variable coeff. Helmholtz operator $-k^{-2}\nabla\cdot(A\nabla) - n$ is not a Fourier multiplier

it is a (semiclassical) pseudo differential operator

generalisation of this thm is the (semiclassical) "elliptic parametrix"