Numerical analysis of the high-frequency Helmholtz eqn: a case study in applied analysis

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Goal of these 3 lectures: illustrate how (relatively simple) tools from (semiclassical) analysis can be used to answer important questions in the numerical analysis of the Helmholtz eqn

A simple proof that the *hp*-FEM does not suffer from the pollution effect for the constant-coefficient full-space Helmholtz equation

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Wavenumber-explicit convergence of the *hp*-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients

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Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element method

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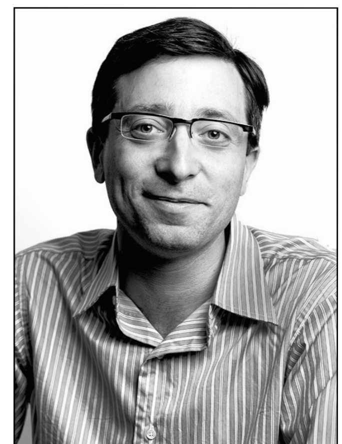
May 3, 2022

The *hp*-FEM applied to the Helmholtz equation with PML truncation does not suffer from the pollution effect

J. Galkowski*, D. Lafontaine†, E. A. Spence‡, J. Wunsch§

July 13, 2022



Today

- model Helmholtz problem
- the NA question
- assemble background results

Model Helmholtz problem

given $f \in L^2_{comp}(\mathbb{R}^d)$ find $u \in H^1_{loc}(\mathbb{R}^d)$ s.t.

$$\begin{cases} k^{-2}\Delta u + u = -f \\ k^{-1}\dfrac{\partial u}{\partial r} - iu = o\left(\dfrac{1}{r^{\frac{d-1}{2}}}\right) \text{ as } r := |x| \to \infty \text{ unif. in } \dfrac{x}{r} \end{cases}$$

$d=3$

$$u(x) = k^{-2}\int_{\mathbb{R}^d} \frac{e^{ik|x-y|}}{4\pi|x-y|} f(y)\,dy$$

$$u(x) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}}\left(F_0\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right)$$

$r \to \infty$

$$\frac{\partial^2 U}{\partial t^2} - c^2\Delta U = F$$

$$U(x,t) = u(x)e^{-i\omega t}, \quad F(x,t) = k^2 f(x) e^{-i\omega t}$$

then u $\quad k = \omega/c$

$$U(x,t) = \frac{e^{ik(r-ct)}}{r^{\frac{d-1}{2}}}(\dots)$$

Variational formulation of model Helmholtz problem

Let $R>0$ s.t. $\operatorname{supp} f \subset B_R$, find $\tilde{u} \in H^1(B_R)$ s.t. $a(\hat{u},v) = F(v)\ \forall v \in H^1(B_R)$

where $a(\hat{u},v) := \int_{B_R} \left(k^{-2}\nabla \hat{u}\cdot\overline{\nabla v} - \hat{u}\,\overline{v}\right) - k^{-1}\langle \mathrm{DtN}\,\hat{u}, v\rangle_{\partial B_R}$

$F(v) := \int_{B_R} f\,\overline{v}$

given $g \in H^{\frac{1}{2}}(\partial B_R)$

let $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B_R})$ be sol^n of

$$(-k^{-2}\Delta - 1)w = 0 \text{ in } \mathbb{R}^d \setminus \overline{B_R}$$

$$w = g \text{ on } \partial B_R$$

$$\mathrm{DtN}\, g := k^{-1}\partial_r w$$

$$\mathrm{DtN} : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$$

Lemma $\tilde{u} = u|_{B_R}$

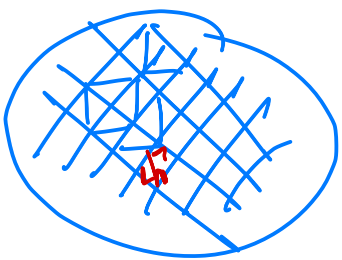
The Galerkin method

$\{\mathcal{H}_N\}_{N=0}^{\infty}$ sequence of f.d. subspaces of $H^1(B_R)$ s.t. $\forall v \in H^1(B_R)$

$$\lim_{N\to\infty}\left(\min_{v_N\in\mathcal{H}_N}\|v-v_N\|_{H^1(B_R)}\right)=0$$

find $u_N \in \mathcal{H}_N$ s.t. $a(u_N, v_N) = F(v_N) \quad \forall v_N \in \mathcal{H}_N$

we'll consider $\mathcal{H}_N$ consisting of piecewise polynomials



hFEM: $h \to 0$ as $N \to \infty$, p fixed

pFEM: $p \to \infty$ as $N \to \infty$, h fixed

hpFEM: $h \to 0$ and $p \to \infty$ as $N \to \infty$

total number of d.o.f. $\sim \left(\frac{p}{h}\right)^d$

goal: "quasi optimality" $\exists\, C_{qo} > 0$ and $N_0 \in \mathbb{N}$ s.t. $\forall N \geq N_0$

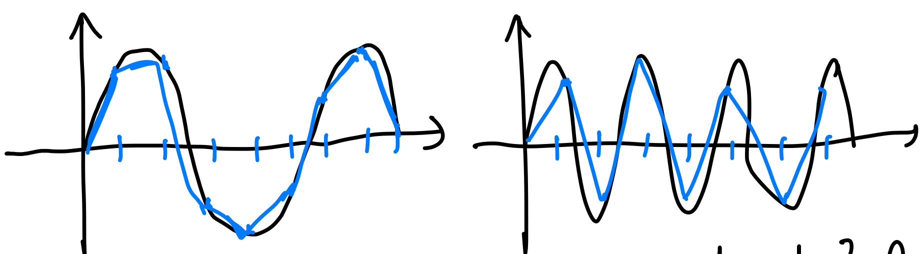
$$\|u-u_N\|_{H^1(B_R)} \leq C_{qo} \min_{v_N\in\mathcal{H}_N}\|u-v_N\|_{H^1(B_R)}$$

$$\|v\|^2_{H^m_k(B_R)} := \sum_{0\leq|\alpha|\leq m}\|(k^{-1}\partial)^\alpha v\|^2_{L^2}$$

The question: for what $h = h(k)$, $p = p(k)$ does quasioptimality hold with $C_{qo} \sim 1$ as $k \to \infty$

Given function oscillating with freq. k, expect $\sim k^d$ d.o.f. for uniform approx as $k \to \infty$, e.g. $h \sim \frac{1}{k}$, p const. $\left(\left(\frac{p}{h}\right)^d = \# \text{dof} \right)$

(cf Nyquist-Shannon-Whittaker)

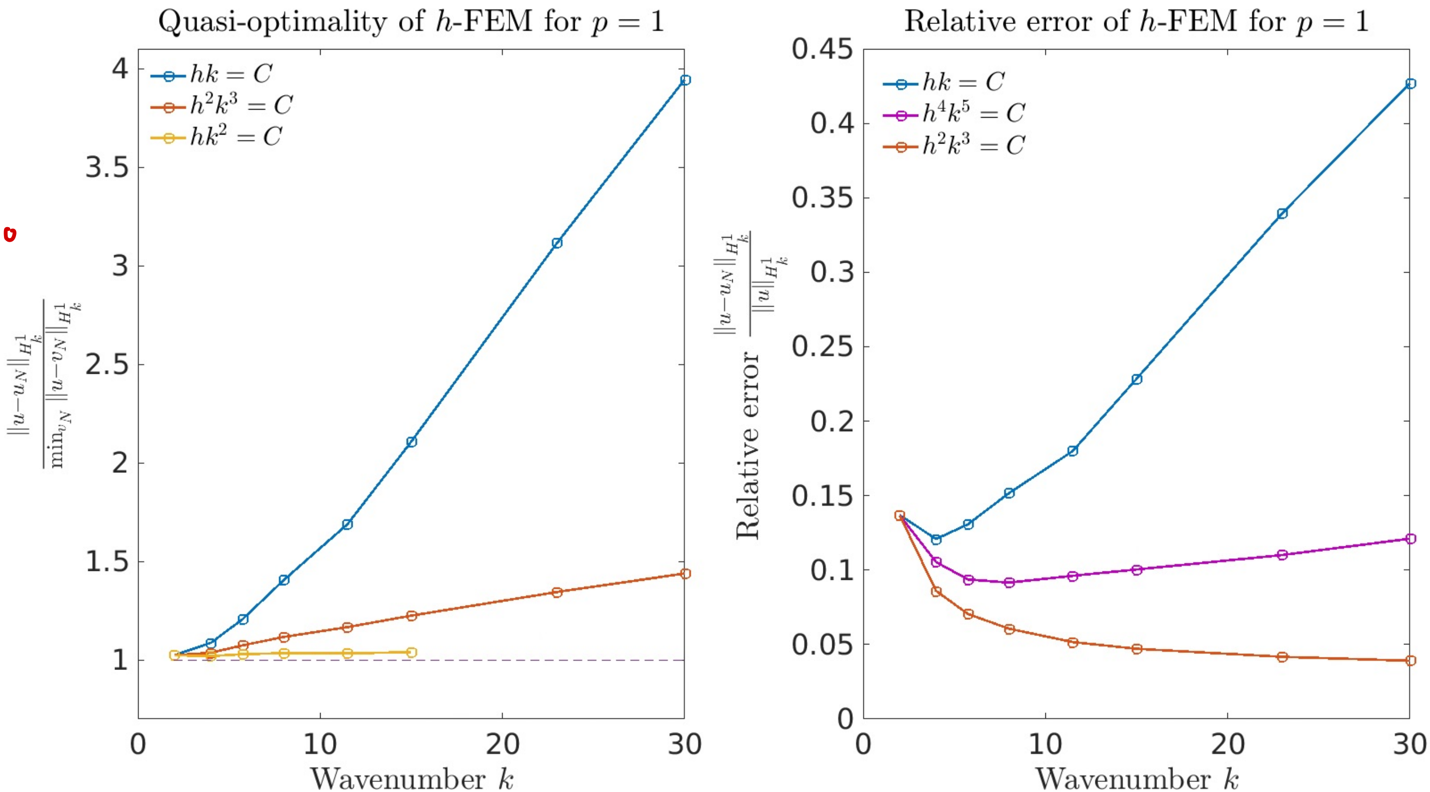


BUT hFEM with $p = 1$ need $h \sim k^{-2}$ for $C_{qo} \sim 1$

$p > 1$ need $h \sim k^{-\left(\frac{p+1}{p}\right)}$ $\longrightarrow$ $\Rightarrow \# \text{dof} \sim \left(k^{\frac{p+1}{p}}\right)^d \gg k^d$

"pollution effect"

C_{10}



Quasi-optimality of h-FEM for $p = 1$
$hk = C$
$h^2k^3 = C$
$hk^2 = C$
$\frac{\|u-u_N\|_{H^1_k}}{\min_{v_N}\|u-v_N\|_{H^1_k}}$
Wavenumber k
Relative error of h-FEM for $p = 1$
$hk = C$
$h^4k^5 = C$
$h^2k^3 = C$
Relative error $\frac{\|u-u_N\|_{H^1_k}}{\|u\|_{H^1_k}}$
Wavenumber k

Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM

F Ihlenburg, I Babuška - Computers & Mathematics with Applications, 1995 - Elsevier

The paper addresses the properties of finite element solutions for the Helmholtz equation. The h-version of the finite element method with piecewise linear approximation is applied to a …

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Finite element solution of the Helmholtz equation with high wave number part II: the hp version of the FEM

F Ihlenburg, I Babuska - SIAM Journal on Numerical Analysis, 1997 - SIAM

In this paper, which is part II in a series of two, the investigation of the Galerkin finite element solution to the Helmholtz equation is continued. While part I contained results on the h …

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$h \sim k^{-\frac{(p+1)}{p}}$

Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation

JM **Melenk**, S **Sauter** - SIAM Journal on Numerical Analysis, 2011 - SIAM

We develop a stability and convergence theory for a class of highly indefinite elliptic boundary value problems (bvps) by considering the Helmholtz equation at high wavenumber k as …

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$k^{-2}\Delta u + u = -f$

Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions

J **Melenk**, S **Sauter** - Mathematics of Computation, 2010 - ams.org

A rigorous convergence theory for Galerkin methods for a model Helmholtz problem in ${\mathbb {R}}^{d} $, $ d\in\{1, 2, 3\} $ is presented. General conditions on the approximation …

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! hp FEM does not suffer from the pollution effect !

if $\frac{hk}{p} \lesssim C_1$, $p \geq C_2 \log k$ then $C_{qo} \sim 1$, $\# \text{dof} \sim \left(\frac{p}{h}\right)^d \sim k^d$

What info do we need about Helmholtz v.f.?

find $u \in H^1(B_R)$ s.t. $a(u,v) = F(v) \quad \forall v \in H^1(B_R)$

- sol$^{\underline{n}}$ exists and is unique

- <u>continuity of $a(\cdot,\cdot)$</u>

  given $k_0 > 0$, $R_0 > 0$ $\exists\, C_{cont} > 0$ s.t. $\forall k \geq k_0$, $\forall R \geq R_0$

$$|a(u,v)| \leq C_{cont} \|u\|_{H^1_k(B_R)} \|v\|_{H^1_k(B_R)} \quad \forall u,v \in H^1(B_R)$$

- <u>Gårding inequality</u>

$$\operatorname{Re} a(v,v) \geq \|v\|^2_{H^1_k(B_R)} - 2\|v\|^2_{L^2(B_R)} \quad \forall v \in H^1(B_R)$$

What info do we need about Helmholtz sol$^{\underline{n}}$ operator?

$$\|u\|_{H^1_k(B_R)} \leq C_{sol} \|f\|_{L^2(B_R)}$$

Thm (Morawetz 1968, 1975)

$$C_{sol} \leq 2kR \sqrt{1 + \left(\frac{d-1}{2kR}\right)^2}$$

multiply $k^{-2}\Delta u + u = -f$ by $x \cdot \nabla u - ik\beta u + \alpha u$ and ibp

Cor given $k_0, R_0 > 0$ $\exists C > 0$ st

$$\|u\|_{H^2_k(B_R)} \leq C\, kR \|f\|_{L^2(B_R)} \quad \forall k \geq k_0,\ R \geq R_0$$